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# *Axiomatic Foundations of Classical Particle Mechanics*[1]

J. C. C. McKINSEY, A. C. SUGAR, & PATRICK SUPPES

Stanford University, Los Angeles, and Stanford University, California

*Communicated by* C. TRUESDELL*

**1. Introduction.** With the publication of Newton's *Principia* in 1687, classical particle mechanics received its first systematic formulation. In the nearly three centuries that have elapsed since that date, the foundations of this discipline have been re-examined and restated a great number of times[2]. From the standpoint of logical rigor and precision, however, none of the existing treatments of classical mechanics seems to be entirely free from serious defects: none of them comes even close to satisfying the standards set, let us say, by Hilbert in his axiomatization of Euclidean geometry[3].

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* The communicator is in complete disagreement with the view of classical mechanics expressed in this article. He agrees, however, that strict axiomatization of general mechanics—not merely the degenerate and conceptually insignificant special case of particle mechanics—is urgently required. While he does not believe the present work achieves any progress whatever toward the precision of the concept of force, which always has been and remains still the central conceptual problem, and indeed the only one not essentially trivial, in the foundations of classical mechanics, he hopes that publication of this paper may arouse the interest of students of mechanics and logic alike, thus perhaps leading eventually to a proper solution of this outstanding but neglected problem.

[1] We are grateful to Professor HERMAN RUBIN, Professor ALFRED TARSKI, and Mr. ROBERT VAUGHT for a number of helpful suggestions in connection with this paper.

[2] Among the works of the eighteenth and nineteenth centuries, those of LAGRANGE, HERTZ, and MACH are outstanding. For more recent works, see MARCOLONGO [1], HAMEL [1], [2], and [3], and SIMON [1]; (the numbers in square brackets refer to items in the Bibliography). The references mentioned are, however, only a small part of the enormous literature in physics, mathematics, and philosophy concerning the foundations of mechanics; in fact, nearly every textbook in the subject attempts something in this direction.

[3] Thus, although HAMEL's formulation of mechanics is rightly regarded as one of the clearest existing treatments of the subject, we find in HAMEL [2] (p. 3), the follow-

An outstanding deficiency of the literature has been the failure to make clear just what the primitive notions of particle mechanics are taken to be. The usual practice has been to omit explicit mention of some of the primitives actually used, and such a practice leads almost inevitably to an insufficient set of axioms[4]. One of our objectives in this paper is to give a set of axioms, based on an explicitly stated list of primitive notions, which we believe are adequate for classical particle mechanics.

It should be remarked that particle mechanics, like almost any other science in deductive form, involves an idealization of actual empirical knowledge—and is thus better conceived as a tool for dealing with the world, than as a picture representing it. We are going to assume, for instance, that time intervals can be arbitrarily small (an assumption for which there is no empirical evidence); and we shall set no positive lower bound to the mass of particles (though there seem to be good empirical grounds for supposing there exists such a bound). We incorporate these idealizations into our system on the pragmatic ground that they simplify the mathematics. We feel, however, that it is very important in science to be aware just which assumptions of this sort one is making, since one might decide later to modify them: hence the desirability of the axiomatic approach.

Any axiomatization of particle mechanics, moreover, must have still an additional arbitrary character, due to the fact that physicists are not quite agreed among themselves as to what is to be meant by a particle. It will not do to say that the concept of a particle is implicitly defined by the axioms of particle mechanics: for the problem is to decide what is the intuitive, informal notion of a particle which is to guide us in setting up such axioms. For example, should small bodies serve as our preliminary model of particles, or should we use centers of mass of bodies as our model? This is no pedantic distinction, for there is an essential difference here when the question of an axiom of impenetrability arises; the first model suggests such an axiom, while the second does not. In deciding such questions we have often been influenced by considerations of convenience and elegance; thus in the above case of the two possible interpretations of particles, for instance, it appears that an axiom of impenetrability does nothing but complicate proofs, and we have accordingly omitted any such axiom.

We believe, nevertheless, that the characterization of classical particle mechanics presented here, while arbitrary in certain ways, does not seriously deviate in any substantive respect from the usual conceptions. Our sole aim has been to present an old subject in a mathematically rigorous way, not to cre-

ing strange axiom: "Die Kräfte $dk$ sind durch ihre 'Ursachen' bestimmt, *d. h.* durch Variable, welche den geometrischen und physikalischen Zustand der umgebenden Materie darstellen. Diese Abhängigkeit ist eindeutig und im allgemeinen stetig und differenzierbar." One does not see how this axiom could intervene in the proofs of theorems, or in the solution of problems.

[4] An axiomatization of mechanics (but of relativistic, rather than classical, mechanics) which avoids this defect, is to be found in HERMES [1]. But HERMES' development appears to be defective in other directions (see ROSSER [1]).

ate a new and unwanted branch of mechanics. Partial evidence for the adequacy of our axioms is given by Theorems 2 and 8. Stronger evidence would be afforded if one could show that a system satisfies our axioms for particle mechanics if and only if it is isomorphic to the system constituted by the centers of mass in some set of rigid bodies; but to do this in an exact way, one would, of course, have first to give an axiomatic foundation of the mechanics of rigid bodies.

Besides our primitive notions and axioms, we shall also avail ourselves of various notions and results of classical mathematics. Thus our system is not completely formalized. But it will be clear how the process of formalization could be completed: it would be necessary merely to add to our primitives the appropriate mathematical and logical notions (real number, addition and multiplication of real numbers, and the like, and sentential connectives, quantifiers, and the like), and to supplement our axioms by the addition of appropriate mathematical and logical axioms and rules.

It will perhaps be helpful to add a word about some of the mathematical notations we shall use, though most of them are standard. We denote the set whose only members are $a_1, a_2, \cdots, a_n$ by $\{a_1, a_2, \cdots, a_n\}$ and the ordered $n$-tuple whose first member is $a_1$, whose second member is $a_2$, and so on, by $\langle a_1, a_2, \cdots, a_n\rangle$.

If $A$ and $B$ are sets, then we denote by $A \times B$ the *Cartesian product* of $A$ and $B$: *i.e.*, the set of all ordered couples $\langle a, b\rangle$, where $a \in A$ and $b \in B$.

We use the word "interval" to mean one-dimensional interval, which can be open, closed, or half-closed, and finite, infinite, or half-infinite: *i.e.*, any connected subset of the real numbers which contains at least two elements.

We use the customary notation for derivatives. When a function is said to have a derivative throughout a closed interval $[a, b]$, we mean that the derivative exists at all interior points of the interval, that the right-hand derivative exists at $a$, and that the left-hand derivative exists at $b$.

By an ($n$-dimensional) vector we mean an ordered $n$-tuple $\langle a_1, \cdots, a_n\rangle$ of real numbers. If $\langle a_1, \cdots, a_n\rangle$ is a vector and $b$ is a real number, then we set

$$b\langle a_1, \cdots, a_n\rangle = \langle a_1, \cdots, a_n\rangle\, b = \langle a_1 b, \cdots, a_n b\rangle.$$

If $\langle a_1, \cdots, a_n\rangle$ and $\langle b_1, \cdots, b_n\rangle$ are two vectors, then we set

$$\langle a_1, \cdots, a_n\rangle + \langle b_1, \cdots, b_n\rangle = \langle a_1 + b_1, \cdots, a_n + b_n\rangle$$

and

$$\langle a_1, \cdots, a_n\rangle - \langle b_1, \cdots, b_n\rangle = \langle a_1 - b_1, \cdots, a_n - b_n\rangle.$$

If $\langle a_1, \cdots, a_n\rangle$ is a vector, we set $-\langle a_1, \cdots, a_n\rangle = \langle -a_1, \cdots, -a_n\rangle$. We shall denote the $n$-dimensional vector all of whose components are 0 by $\bar{0}_n$.

We call two vectors $\alpha$ and $\beta$, neither of which is $\bar{0}_n$, *parallel* if there exists a real number $k$ such that $\alpha = k\cdot\beta$.

If $\alpha = \langle a_1, \cdots, a_n \rangle$ is any vector, we set $|\alpha| = \sqrt{a_1^2 + \cdots + a_n^2}$.

If, for every $t$ in an interval $T$ the (numerical valued) functions $f_1(t), \cdots, f_n(t)$ possess derivatives with respect to $t$, then we set

$$\frac{d}{dt}\langle f_1(t), \cdots, f_n(t)\rangle = \left\langle \frac{d}{dt} f_1(t), \cdots, \frac{d}{dt} f_n(t)\right\rangle.$$

We define higher derivatives of vector functions in an analogous way.

If each of the $n$ infinite series

$$\begin{array}{c} x_{11} + x_{12} + \cdots \\ x_{21} + x_{22} + \cdots \\ \cdot \\ \cdot \\ \cdot \\ x_{n1} + x_{n2} + \cdots \end{array} \tag{1}$$

is convergent, and if the series converge respectively to $x_1, x_2, \cdots, x_n$, then we say that the series of vectors

$$\langle x_{11}, x_{21}, \cdots, x_{n1}\rangle + \langle x_{12}, x_{22}, \cdots, x_{n2}\rangle + \cdots \tag{2}$$

is convergent, and we set

$$\langle x_1, \cdots, x_n\rangle = \langle x_{11}, \cdots, x_{n1}\rangle + \langle x_{12}, \cdots, x_{n2}\rangle + \cdots .$$

If each of the series (1) is absolutely convergent, we also call (2) absolutely convergent.

**2. Primitive notions.** Our system of particle mechanics is based on five primitive notions: $P$, $T$, $m$, $s$, and $f$. $P$ and $T$ are sets, $m$ is a unary function, $s$ is a binary function, and $f$ is a ternary function.

The intended physical interpretation of $P$ is as the set of particles. It must be borne in mind, however, that, because of the abstract character of our axioms, we are not restricted to physical models; thus in some cases it may be useful to interpret $P$ as a set of numbers, let us say, or a set of sets, or a set of functions over some domain.

$T$ is to be interpreted physically as a set of real numbers measuring elapsed times (in terms of some unit of time, and measured from some origin of time). Thus we leave aside all problems connected with the measurement of time (as well as of mass, distance, and force); this is not because we believe all such problems have been solved—or that they are unimportant—but merely because we consider it is possible to separate mechanics proper from such epistemological and experimental questions.[5]

[5] We may add that the attempt to deal with both sorts of problems simultaneously is in our opinion responsible for much of the confusion and murkiness characteristic of the usual discussions of the foundations of physics.

If $p$ is a member of $P$ (that is to say, in the physical interpretation, if $p$ is a particle), then $m(p)$ is to be interpreted physically as the numerical value of the *mass* of $p$.

It would be possible to generalize our system, by regarding mass as a function also of time; and such a generalized system could find useful applications, for example, to the theory of rockets (where, as the fuel is used up, the mass of the rocket decreases). But we have preferred to keep close to the traditional expositions—in which mass has been regarded as invariant over time.

If $p$ is in $P$, and $t$ is in $T$, then $s(p, t)$ is an $n$–dimensional vector. For $n = 3$ (or for $n < 3$, if we are concerned with plane particle mechanics, or with one–dimensional particle mechanics) $s(p, t)$ is to be interpreted physically as a vector giving the *position* of $p$ at time $t$. Although the most obviously useful physical interpretations of our system are obtained with $n \leqq 3$, we have left $n$ arbitrary because none of the theorems we prove depends on the value of $n$.

It should be noticed that the primitive $s$ fixes the choice of a coordinate system. It would also be possible to develop particle mechanics taking as a primitive the set of all admissible (inertial) coordinate systems:[6] that is to say, roughly speaking, the class of all coordinate systems with respect to which the particles in question satisfy Newton's Second Law.

The intended physical meaning of our last privitive is slightly more complicated. In dealing with an empirical situation to which we wish to apply the theory of particle mechanics, each particle is ordinarily subjected to a number of different forces. It is true that the motion of the particle is determined solely by the resultant of these forces; but in most cases the problem is originally stated in terms of a variety of forces. In order to retain a reasonable amount of flexibility in our system, it is thus desirable to allow for the action of a number of forces on each particle; however, we assume that at most a countable infinity of forces act on a given particle. If $p$ is a particle, therefore, and $t$ is any element of $T$, we might denote these forces by: $F_1(p, t)$, $F_2(p, t)$, $\cdots$ . But, in order to avoid the cumbrousness of having an infinity of primitive functions in the system, we denote these forces by: $f(p, t, 1)$, $f(p, t, 2)$, $\cdots$ . Thus if $p$ is any member of $P$, if $t$ is any member of $T$, and if $i$ is any member of the set $I$ of positive integers, then $f(p, t, i)$ is a vector giving the components (parallel to the axes of the coordinate system) of the $i^{th}$ *force* acting on the particle $p$ at the time $t$. (It is clear that the ordering of the forces can be arbitrary: thus it is not of any interest that $f(p, t, 1)$ is the first, and $f(p, t, 2)$ is the second, force acting on $p$ at time $t$; it is important only that $f(p, t, 1)$ and $f(p, t, 2)$ may be distinct.)

We make throughout the assumption, which is customary in modern mathematics, that the primitive functions are not defined over any sets larger than those mentioned in the axioms. Thus $m(p)$ is defined only when $p$ is in $P$; similarly $s(p, t)$ is defined only when $p$ is in $P$ and $t$ is in $T$; and $f(p, t, i)$ is defined only when $p$ is in $P$, $t$ is in $T$, and $i$ is in $I$. Hence the function $m$, for example, is a certain set of ordered couples of the form $\langle p, x\rangle$, where $p$ belongs to $P$ and $x$ is

[6] Such a procedure is used by HERMES for relativistic mechanics; see HERMES [1].

a real number. It follows that the primitives $P$ and $T$ are actually definable:[7] thus we could define $P$ to be the set of the first elements of the couples constituting $m$. In order to avoid an appearance of strangeness in our axiom system, we have nevertheless retained $P$ and $T$ as primitive notions. If the system were to be completely formalized (*i.e.*, embedded, together with the necessary parts of mathematics, in a system of formal logic), then it would doubtless be better to eliminate the notions $P$ and $T$ altogether; but it should be obvious what changes in the axiom system this would involve.

**3. Axioms.** Using the five primitive notions which were intuitively characterized in the previous section, we now give our axioms for classical particle mechanics. Some remarks in justification of the axioms, and some suggestions for alternative axiomatizations, will be given below.

A system $\Gamma = \langle P, T, m, s, f\rangle$ which satisfies Axioms $P1$–$P6$ is called an *n–dimensional system of particle mechanics* (or sometimes, when there is no danger of ambiguity, simply a *system of particle mechanics*).

### *Kinematical Axioms*

*Axiom P*1. *P is a non-empty, finite set.*

*Axiom P*2. *T is an interval of real numbers.*

*Axiom P*3. *If p is in P and t is in T, then $s(p, t)$ is an n–dimensional vector such that $d^2/dt^2\, s(p, t)$ exists.*

### *Dynamical Axioms*

*Axiom P*4. *If p is in P, then $m(p)$ is a positive real number.*

*Axiom P*5. *If p is in P and t is in T, then $f(p, t, 1), f(p, t, 2), \cdots, f(p, t, i), \cdots$ are n–dimensional vectors such that the series $\sum_{i=1}^{\infty} f(p, t, i)$ is absolutely convergent.*

*Axiom P*6. *If p is in P and t is in T, then*

$$m(p) \frac{d^2}{dt^2} s(p, t) = \sum_{i=1}^{\infty} f(p, t, i).$$

The condition, in Axiom $P1$, that $P$ is non-empty, could be omitted without any serious modification of the system; there seems to be little reason for considering empty sets of particles, however, so we have not made such a generalization.

The condition that $P$ is finite is put in to make our system agree with the usual formulations of particle mechanics. In this connection it should be noticed that, if we want to be able to interpret particles as centers of mass of rigid bodies, and if we suppose that rigid bodies always have non-zero volume, and cannot penetrate each other, then at least the possibility is excluded that there be an uncountable infinity of particles. But an essential generalization of our axiom

[7] We shall see in Section 5, on the other hand, that $m$, $s$, and $f$ are mutually independent.

system would be obtained if we were to replace $P1$ by the axiom: "$P$ is non-empty, and either finite or countably infinite." If Axiom $P1$ were to be liberalized in this way, however, then it would probably be desirable to add some additional axioms, so as to insure that the total mass and kinetic energy of the system be finite.

Referring back to the explanation in Section 1 of the term "interval" we see that Axiom $P2$ means simply that $T$ is a connected set of real numbers (containing at least two members). It might be thought that it would be preferable to suppose that $T$ is always the set of all real numbers (in which case the primitive notion $T$, as well as Axiom $P2$, could be omitted). This has the disadvantage, however, that in many applications it would be unnatural or inconvenient to suppose that the motion of a system of particles persists throughout all time. Thus if we wish to study the trajectory of a cannon ball, it seems very natural to suppose that $T$ starts at the moment the ball leaves the muzzle of the cannon. Or consider the case of two particles being pulled together by their mutual gravitational attraction; here it is convenient to suppose that $T$ is an open interval, and that the right-hand end-point of $T$ is the instant of contact of the two particles; since the acceleration of the particles approaches infinity as the particles approach contact, we cannot take $T$ to include the moment of contact without introducing discontinuities into the accelerations. (To treat any larger interval of time, we would need to resort to a different branch of mechanics—which might be called "impact mechanics.")

The condition in our Axiom $P3$, that the function $s$ be twice differentiable, is slightly more restrictive than that imposed by an axiom of Hamel, who requires (see Hamel [**2**], p. 2) merely that $s$ be piecewise twice differentiable. Our stronger axiom makes it easier to prove general theorems, however, and we believe that most applications can still be taken care of by the device of considering several successive realizations of the axioms instead of a single one. Thus suppose that an object slides across the surface of a smooth horizontal table with a constant velocity, and then falls off the edge. Here the second derivative of the position vector (*i.e.*, the acceleration) does not exist at the instant when the object is just at the edge of the table. On the other hand, at that instant the left-hand and right-hand second derivatives both exist (the first is zero, and the second is the acceleration of gravity). Hence we can deal with this situation by considering two realizations of our axiom system, with two consecutive intervals $T_1$ and $T_2$, of time: $T_1$ lasts from the start of the motion till the object reaches the edge of the table; and $T_2$ begins with the object leaving the edge of the table, and lasts till the end of the motion.

In Axiom $P4$ we have taken mass to be always positive. The apparently harmless generalization of allowing mass sometimes to be zero would have the following inconvenient consequence: if $m(p) = 0$, then Axiom $P6$ would impose no condition on the acceleration of the particle, and hence its motion would be quite indeterminate. Simon (see Simon [**1**]) permits mass even to be negative;

but this generalization does not appear to be very useful so far as regards applications.

Axiom $P5$ imposes the condition that the sum of the forces acting on a particle be absolutely convergent. This is done in order that the motion of a particle be independent of the order of naming the forces applied to it.

Axiom $P6$ contains a formulation of Newton's Second Law. This is the only one of our axioms which is ordinarily explicitly stated. The kinematical axioms, in particular, are practically never mentioned. Actually, however, $P6$ does not by itself constitute a sufficient basis for classical particle mechanics: if we combine $P3$ and $P6$ into a single axiom, the resulting five axioms are mutually independent. Moreover, it should be apparent from the discussion above that it is not altogether beyond question what other axioms besides $P6$ should be assumed: to state the other axioms explicitly is to begin to consider various alternatives.

We close this section with a few remarks about some axioms which might have been assumed, but were not.

In the first place, as was pointed out in Section 1, we do not assume such an Axiom of Impenetrability as the following: "If $p_1$ and $p_2$ are distinct members of $P$, and if $t$ is in $T$, then $s(p_1, t) \neq s(p_2, t)$." This is because we have in mind interpreting the elements of $P$ as the centers of mass of rigid bodies; and it can easily happen (as sometimes, for example, when a bullet is fired through the hole in a doughnut) that the centers of mass of distinct bodies may at a certain moment coincide.

Secondly, we have not assumed Newton's First Law; but this is a trivial consequence of our axioms (see Theorem 1).

Finally, we have not taken the Third Law as an axiom. We have omitted this law because it often happens in applications that one wants to consider a system of particles where it is *not* true that to every action there is an equal and opposite reaction: thus in exterior ballistics, for example, we never consider the perturbation of the earth's orbit occasioned by firing a cannon. The absence of this law as an axiom, however, is counterbalanced by the fact that we can prove Theorem 8, which amounts to saying that every model of our axioms can be embedded in a model which also satisfies Newton's Third Law.

Closely connected with the Third Law is the question of the distinction between internal and external forces. This distinction could have been made initially by replacing the primitive $f$ by two new primitives: $g$, to represent external forces; and $h$, to represent internal forces. If $p$ is in $P$, $t$ is in $T$, and $i$ is in $I$, then $g(p, t, i)$ is to be interpreted physically as the $i^{th}$ external force acting on $p$ at time $t$. If $p$ and $q$ are in $P$, $t$ is in $T$, and $i$ is in $I$, then $h(p, q, t, i)$ is to be interpreted physically as the $i^{th}$ internal force which the particle $q$ exerts on $p$ at time $t$. The revised axioms would be the following: $P1$, $P2$, $P3$, $P4$, as before; two axioms on $g$ and $h$, similar to $P5$ on $f$; a seventh axiom, similar to $P6$, expressing Newton's Second Law; an eighth axiom asserting that, for $p$ and $q$ in $P$, for $t$ in $T$, and for $i$ in $I$, $h(p, q, t, i) = -h(q, p, t, i)$; and finally a ninth axiom asserting that, for $p$ and $q$ in $P$, for $t$ in $T$, and for $i$ in $I$, the three vectors

$s(p, t) - s(q, t)$, $h(p, q, t, i)$, and $h(q, p, t, i)$, unless one of them is $\bar{0}_n$, are all parallel.[8]

We have chosen to use the single primitive $f$, however, instead of the two primitives $g$ and $h$ for two reasons. In the first place, to incorporate such a distinction into the axiom system increases the difficulty of proving theorems of an abstract algebraic character (such as Theorems 4, 5, 6, 7, and 8). Secondly, we are not of the opinion this change in our primitives would serve any very useful purpose. This is chiefly for the reason that the usual notion of an internal force is such that not every pair of balanced forces is regarded as internal; a ball hanging at rest on a string, for example, is acted on by two balanced forces (the attraction of the earth and the tension of the string), but these forces would ordinarily both be called external. Instead of introducing the new primitives $g$ and $h$, we believe that it is formally more simple and useful to speak of sets of balanced pairs of forces, instead of speaking of internal forces. As a matter of fact, many formulations of general dynamics do not depend in any way on a distinction between internal and external forces, or on the notion of one particle acting on another.[9]

We now make these notions more precise by introducing some formal definitions.

***Definition*** 1. Let $\langle P, T, m, s, f\rangle$ be an $n$-dimensional system of particle mechanics, let $p$ and $q$ be members of $P$, and let $i$ and $j$ be members of $I$. Then we say that the $i^{th}$ *force acting on* $p$ *and the* $j^{th}$ *force acting on* $q$ *balance each other* if the following conditions are satisfied for every $t$ in $T$:

(1) $f(p, t, i) = -f(q, t, j)$

(2) the vectors $f(p, t, i)$, $f(q, t, j)$, and $s(p, t) - s(q, t)$, unless one of them is $\bar{0}_n$, are all parallel.

***Definition*** 2. Let $\langle P, T, m, s, f\rangle$ be a system of particle mechanics, and let

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[8] The eighth and ninth axioms require that the internal forces satisfy Newton's Third Law. The eighth axiom corresponds to what HAMEL ([**2**], p. 25) calls the *first complete reaction principle*, and the ninth axiom to what he calls the *second complete reaction principle* for particle mechanics. The formulation of classical particle mechanics given by these nine axioms corresponds closely to the unaxiomatized, intuitive one given by Joos ([**1**], Chap. VI), and by BANACH ([**1**], Chap. V). However, neither Joos nor BANACH permits a particle to exert more than one force on another; in addition, both of these authors lump all external forces together and introduce a notation only for the *resultant* external force. HAMEL ([**2**], p. 25) gives axioms which are more restrictive than these. If there are $k$ particles in a system, he permits to act on a given particle only the $k - 1$ internal forces due to the other particles. On the other hand, APPELL ([**1**], p. 143) and GRAMMEL ([**1**], p. 340) require the eighth axiom but not the ninth—that is, they do not require that the force exerted by one particle on another be directed along the line joining the two particles.

These brief comparative remarks indicate that there is no precise agreement concerning exactly what the assumptions of classical particle mechanics are. This difference of opinion concerning the more restrictive kinds of assumptions, of which the ninth axiom is an example, is a partial justification for the more general axioms $P1$–$6$.

[9] Such general formulations are to be found, for instance, in OSGOOD ([**1**], Appendix D) and MARCOLONGO ([**1**], vol. 2, Chap. III).

$P \times I$ be the Cartesian product of $P$ and $I$: i.e., the set of all ordered couples $\langle p, i\rangle$ such that $p \epsilon P$ and $i \epsilon I$. Then we call a subset $A$ of $P \times I$ a *balanced set*, if there exist two mutually exclusive sets $A_1$ and $A_2$, whose union is $A$, and a one-to-one correspondence between $A_1$ and $A_2$ such that, whenever $\langle p, i\rangle$ in $A_1$ corresponds to $\langle q, j\rangle$ in $A_2$, then the $i^{th}$ force acting on $p$ and the $j^{th}$ force acting on $q$ balance each other.

*Remark.* It might be thought that instead of giving the above two definitions, one could simply define directly a "set of balanced forces." The inadequacy of such a procedure, however, can be seen from the fact that distinct members of $P \times I$ can be associated with the same force vector. Thus suppose that $P$ consists of just three elements, $p_1$, $p_2$, and $p_3$ each having a mass of 1; and that, for all $t$ in $T$, $s(p_1, t) = \langle t^2 + 2, 0\rangle$, $s(p_2, t) = \langle -t^2, 0\rangle$, and $s(p_3, t) = \langle -t^2 + 1, 0\rangle$, that $f(p_1, t, 1) = \langle 2, 0\rangle$, that $f(p_2, t, 1) = f(p_3, t, 1) = \langle -2, 0\rangle$, and that, for $i > 1$, $f(p_1, t, i) = f(p_2, t, i) = f(p_3, t, i) = \langle 0, 0\rangle$. Then if one speaks of the set $\{\langle 2, 0\rangle, \langle -2, 0\rangle\}$ as a set of balanced forces, it is not clear whether the force $\langle -2, 0\rangle$ is to be construed as the first force acting on $p_2$, or as the first force acting on $p_3$.

For later purposes we shall need the notion of a Newtonian system, and it is convenient to define the term at this point.

***Definition*** 3. A system $\langle P, T, m, s, f\rangle$ is called *Newtonian* if the set $P \times I$ is balanced.

**4. Theorems.** While it would be inappropriate to take up a great deal of space proving the many propositions of particle mechanics usually stated in the textbooks, we shall give in this section some theorems which afford evidence that our axioms are actually adequate for particle mechanics—as well as some theorems which are interesting from an algebraic standpoint. Some of the easier proofs have been omitted.

Our first theorem, as was mentioned earlier, is a formulation of Newton's First Law.

***Theorem*** 1. *Let* $\Gamma = \langle P, T, m, s, f\rangle$ *be an n-dimensional system of particle mechanics, and let p be a member of P such that, for all t in T,*

$$\sum_{i=1}^{\infty} f(p, t, i) = \bar{0}_n .$$

*Then there are n-dimensional vectors* $\alpha$ *and* $\beta$ *such that, for all t in T,*

$$s(p, t) = \alpha + \beta t.$$

*Proof.* By Axioms $P3$, $P4$, and $P6$.

The following theorem amounts to saying that the whole history of a system of particle mechanics is determined by $P$, $T$, $m$, $f$, and appropriate initial conditions. In axiomatizing any branch of deterministic physics it is important to be able to prove such a theorem, since it gives partial evidence of the adequacy of the axioms; in the case of our system, the proof is so easy that we have omitted it.

***Theorem* 2.** *Let $P$ and $T$ be sets, and $m$ and $f$ functions, which satisfy Axioms $P1$, $P2$, $P4$, $P5$, and suppose, in addition, that for every $p$ in $P$ there exists a vector-valued function $g_p$ such that*

$$\frac{d}{dt}[g_p(t)] = \sum_{i=1}^{\infty} f(p, t, i).$$

*Let $P = \{p_1, \cdots, p_r\}$; let $t_0$ be a fixed element of $T$, and let $\alpha_1, \cdots, \alpha_r$ and $\beta_1, \cdots, \beta_r$ be $2r$ fixed vectors. Then there exists one and only one function $s$ such that $\langle P, T, m, s, f\rangle$ is a system of particle mechanics, and such that, for $i = 1, \cdots, r$,*

$$s(p_i, t_0) = \alpha_i \qquad \text{and} \qquad \frac{d}{dt} s(p_i, t)\Big|_{t=t_0} = \beta_i.$$

In connection with a certain class of theorems a problem of some philosophical interest arises. We have in mind those theorems which are ordinarily stated in the subjunctive mood as contrary-to-fact conditionals. Recent philosophical literature[10] indicates how unsatisfactory an analysis we yet have of such assertions, and it consequently seems desirable to eliminate all use of them in a precisely formulated axiomatization of mechanics. We show how this may be done for one of the most familiar of these theorems: namely, the assertion that the center of mass of a system of particles moves *as if* all the mass *were concentrated* there and the resultant of all the "external" forces *acted* there. Without using the subjunctive mood this theorem can be formulated as follows.

***Theorem* 3.** *Let $\langle P, T, m, s, f\rangle$ be a system of particle mechanics, let $A$ be any balanced subset of $P \times I$, let $b$ be any object, and let*

$$(i) \qquad P' = \{b\}$$

$$(ii) \qquad m'(b) = \sum_{p \in P} m(p)$$

$$(iii) \qquad s'(b, t) = \frac{1}{\sum_{p \in P} m(p)} \left[\sum_{p \in P} m(p) s(p, t)\right]$$

$$(iv) \qquad f'(b, t, 1) = \sum_{\langle p, i\rangle \notin A} f(p, t, i)$$

$$(v) \qquad f'(b, t, i) = \bar{0}_n \quad \text{for} \quad i > 1.$$

*Then $\langle P', T', m', s', f'\rangle$ is a system of particle mechanics.*

We remark that the above theorem depends upon the fact that $P$ is a finite set; if $P$ were not finite, *(iii)* and *(iv)* would not suffice to define $s'$ and $f'$.

Now we turn to some theorems about systems of particle mechanics which are analogous to certain familiar theorems of modern algebra. In particular, we

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[10] See GOODMAN [1], and pp. 14–15 of QUINE [1].

examine some of the ways in which it is possible to construct new systems of particle mechanics from given ones.

***Definition*** 4. If $\langle x_1, \cdots, x_r\rangle$ and $\langle y_1, \cdots, y_s\rangle$ are vectors (not necessarily having the same number of components), we set $\langle x_1, \cdots, x_r\rangle \oplus \langle y_1, \cdots, y_s\rangle = \langle x_1, \cdots, x_r, y_1, \cdots, y_s\rangle$. If $f$ is a function defined over some domain $D$, and assuming $r$–dimensional vectors as values, and if $g$ is a function defined over the same domain $D$, and assuming $s$–dimensional vectors as values, then by $f \oplus g$ we mean the function $h$ such that, for every $x$ in $D$, $h(x) = f(x) \oplus g(x)$. (Thus, if $f(1) = \langle 1, 2\rangle$ and $g(1) = \langle 2, 3, 4\rangle$, then $(f \oplus g)$ $(1) = \langle 1, 2, 2, 3, 4\rangle$.) If $\Gamma = \langle P, T, m, s, f\rangle$ and $\Gamma' = \langle P, T, m, s', f'\rangle$ are systems of particle mechanics, then by $\Gamma \oplus \Gamma'$ we mean the system $\langle P, T, m, s \oplus s', f \oplus f'\rangle$; we call $\Gamma \oplus \Gamma'$ the *concatenation* of $\Gamma$ and $\Gamma'$.

***Theorem*** 4. *If $\Gamma = \langle P, T, m, s, f\rangle$ is an $r$–dimensional system of particle mechanics, and $\Gamma' = \langle P, T, m, s', f'\rangle$ is an $s$–dimensional system of particle mechanics, then $\Gamma \oplus \Gamma'$ is an $(r + s)$–dimensional system of particle mechanics.*

***Theorem*** 5. *If $\Gamma$ is any $n$–dimensional system of particle mechanics, then there exist $n$ uniquely determined one–dimensional systems of particle mechanics, $\Delta_1$, $\Delta_2, \cdots, \Delta_n$, such that $\Gamma = \Delta_1 \oplus \Delta_2 \oplus \cdots \oplus \Delta_n$. If $r$ and $s$ are positive integers whose sum is $n$, then there exists exactly one $r$–dimensional system of particle mechanics, $\Gamma_r$, and exactly one $s$–dimensional system of particle mechanics, $\Gamma_s$, such that $\Gamma = \Gamma_r \oplus \Gamma_s$.*

We notice that the operation of concatenation has certain properties in common with the operation of forming the direct union of two algebras. It is because of this analogy that it sometimes suffices, in order to prove a theorem about 3–dimensional mechanics, to prove it about one–dimensional mechanics; this device has often been used in intuitive developments of mechanics.

It is worthy of remark that Theorems 4 and 5 would not necessarily still remain true if we were to strengthen our axioms in various ways. In order to make this point clear, we define now a special class of systems of particle mechanics; examples of systems satisfying the conditions of this definition are to be found in the domains of celestial mechanics and electrostatics.

***Definition*** 5. Let $\Gamma = \langle P, T, m, s, f\rangle$ be an $n$–dimensional system of particle mechanics, where $P = \{p_1, \cdots, p_r\}$. Then we call $\Gamma$ an *ultra-classical* system if there exist $r^2$ functions $\theta_{i,j}$ (for $i, j = 1, \cdots, r$), each of which is a real-valued function of a real variable, and such that, for all $t$ in $T$,

(*i*) $\theta_{i,j} = \theta_{j,i}$;

(*ii*) $f(p_i, t, j) = \theta_{i,j}(\,|\,s(p_i, t) - s(p_j, t)\,|\,)[s(p_i, t) - s(p_j, t)]$, for $i, j = 1, \cdots, r$;

(*iii*) $f(p_i, t, j) = \bar{0}_n$ for $i = 1, \cdots, r$, and $j > r$.

From this definition it is immediately clear that every ultra-classical system is Newtonian.

It is easy to construct two ultra-classical systems whose concatenation is not ultra-classical; moreover, not every two–dimensional ultra-classical system can be represented as a concatenation of one–dimensional ultra-classical systems.

It is of interest to notice, however, that Theorem 5 remains true if we replace "system" by "Newtonian system." But the concatenation of two Newtonian systems is not necessarily Newtonian.

We turn now to a notion which is analogous to the notion of a subalgebra of an algebra.

***Definition*** 6. Let $\Gamma = \langle P, T, m, s, f\rangle$ be a system of particle mechanics; let $P'$ be a non-empty subset of $P$; and let $m'$, $s'$, and $f'$ be the functions $m$, $s$, and $f$ with their first arguments restricted to $P'$: thus $m'$, for example, is defined only over $P'$, and, for all $p$ in $P'$, $m'(p) = m(p)$. Then we call $\Gamma' = \langle P', T, m', s', f'\rangle$ a *subsystem* of $\Gamma$.

***Theorem*** 6. *Every subsystem of a system of particle mechanics is again a system of particle mechanics.*

*Remark.* We notice, on the other hand, that a subsystem of a Newtonian system is not necessarily Newtonian; nor is a sub-system of an ultra-classical system necessarily ultra-classical.

***Definition*** 7. Two systems $\langle P, T, m, s, f\rangle$ and $\langle P', T', m', s', f'\rangle$ of particle mechanics are called *disjoint* if $P$ and $P'$ are mutually exclusive.

***Definition*** 8. Let $\Gamma = \langle P, T, m, s, f\rangle$ and $\Gamma' = \langle P', T', m', s', f'\rangle$ be disjoint systems of particle mechanics such that $T' = T$. Then by the *sum* of $\Gamma$ and $\Gamma'$, which we denote by $\Gamma + \Gamma'$, we mean the system $\Gamma'' = \langle P'', T'', m'', s'', f''\rangle$ where:

(*i*) $T'' = T$;

(*ii*) $P''$ is the set-theoretical sum of $P$ and $P'$;

(*iii*) If $p$ is in $P$, then, for every $t$ in $T$, and for every positive integer $i$,

$$m''(p) = m(p)$$

$$s''(p, t) = s(p, t)$$

$$f''(p, t, i) = f(p, t, i);$$

(*iv*) If $p$ is in $P'$, then, for every $t$ in $T$, and for every positive integer $i$,

$$m''(p) = m'(p)$$

$$s''(p, t) = s'(p, t)$$

$$f''(p, t, i) = f'(p, t, i).$$

***Theorem*** 7. *If* $\Gamma = \langle P, T, m, s, f\rangle$ *and* $\Gamma' = \langle P', T', m', s', f'\rangle$ *are disjoint n–dimensional systems of particle mechanics such that* $T' = T$, *then* $\Gamma + \Gamma'$ *is also a system of particle mechanics. Moreover* $\Gamma$ *and* $\Gamma'$ *are subsystems of* $\Gamma + \Gamma'$.

The following theorem,[11] as was mentioned earlier, constitutes our justification for not including Newton's Third Law as an axiom.

[11] We are indebted to Professor HERMAN RUBIN for an improvement in the formulation of this theorem.

Historically, the theorem is reminiscent of HERTZ's use of "concealed" particles to embed every system of particles in a "free" system obeying his fundamental Principle of the Straightest Path. See HERTZ [1], Chap. V.

***Theorem* 8.** *Every system of particle mechanics is a subsystem of a Newtonian system.*

*Proof.* We shall carry out the proof only for the special case of a 2-dimensional system containing just one particle; it should be obvious what changes would be necessary to take care of the general case.

We begin with a few words explaining the intuitive idea back of this proof. Suppose, for simplicity, we are given a particle $p_1$ in the first quadrant with arbitrary forces acting on it. We place a particle "on top of" $p_1$, and three particles in the other three quadrants placed symmetrically with respect to the $x$- and $y$-axes. It is then possible to introduce forces on the four new particles so as to make the resulting system Newtonian.

We now turn to the formal proof.

It is convenient to introduce two functions each of which singles out a component of a two-dimensional vector. If $\langle x_1, x_2\rangle$ is any two-dimensional vector, we set $\phi_1(\langle x_1, x_2\rangle) = x_1$ and $\phi_2(\langle x_1, x_2\rangle) = x_2$.

Let $\Gamma_1 = \langle\{p_1\}, T, m_1, s_1, f_1\rangle$ be a 2-dimensional system of particle mechanics. Let $\Gamma_2 = \langle\{p_2\}, T, m_2, s_2, f_2\rangle$ be related to $\Gamma_1$ as follows:

$$p_2 \neq p_1$$

$$m_2(p_2) = m_1(p_1)$$

$$s_2(p_2, t) = s_1(p_1, t), \quad \text{for every} \quad t \quad \text{in} \quad T,$$

$$f_2(p_2, t, 1) = \left\langle 2\phi_1\left[\sum_{i=1}^{\infty} f_1(p_1, t, i)\right], 0\right\rangle$$

$$f_2(p_2, t, 2) = \left\langle 0, 2\phi_2\left[\sum_{i=1}^{\infty} f_1(p_1, t, i)\right]\right\rangle$$

$$f_2(p_2, t, i + 2) = -f_1(p_1, t, i) \quad \text{for} \quad i = 1, 2, \cdots.$$

Let $\Gamma_3 = \langle\{p_3\}, T, m_3, s_3, f_3\rangle$ be a two-dimensional system related to $\Gamma_1$ and $\Gamma_2$ as follows:

$$p_3 \notin \{p_1, p_2\}$$

$$m_3(p_3) = 2m_1(p_1)$$

$$s_3(p_3, t) = \langle -\phi_1[s_1(p_1, t)], \phi_2[s_1(p_1, t)]\rangle$$

$$f_3(p_3, t, 1) = \left\langle -2\phi_1\left[\sum_{i=1}^{\infty} f_1(p_1, t, i)\right], 0\right\rangle$$

$$f_3(p_3, t, 2) = \left\langle 0, 2\phi_2\left[\sum_{i=1}^{\infty} f_1(p_1, t, i)\right]\right\rangle$$

$$f_3(p_3, t, i) = \langle 0, 0\rangle \quad \text{for} \quad i > 2.$$

Now let $\Gamma_4 = \langle\{p_4\}, T, m_4, s_4, f_4\rangle$ be a two-dimensional system such that

$$p_4 \notin \{p_1, p_2, p_3\}$$

$$m_4(p_4) = 2m_1(p_1)$$

$$s_4(p_4, t) = \langle -\phi_1[s_1(p_1, t)], -\phi_2[s_1(p_1, t)]\rangle$$

$$f_4(p_4, t, 1) = \left\langle -2\phi_1\left[\sum_{i=1}^{\infty} f_1(p_1, t, i)\right], 0\right\rangle$$

$$f_4(p_4, t, 2) = \left\langle 0, -2\phi_2\left[\sum_{i=1}^{\infty} f_1(p_1, t, i)\right]\right\rangle$$

$$f_4(p_4, t, i) = \langle 0, 0\rangle \quad \text{for} \quad i > 2.$$

Finally, let $\Gamma_5 = \langle\{p_5\}, T, m_5, s_5, f_5\rangle$ be a two-dimensional system such that

$$p_5 \notin \{p_1, p_2, p_3, p_4\}$$

$$m_5(p_5) = 2m_1(p_1)$$

$$s_5(p_5, t) = \langle \phi_1[s_1(p_1, t)], -\phi_2[s_1(p_1, t)]\rangle$$

$$f_5(p_5, t, 1) = \left\langle 2\phi_1\left[\sum_{i=1}^{\infty} f_1(p_1, t, i)\right], 0\right\rangle$$

$$f_5(p_5, t, 2) = \left\langle 0, -2\phi_2\left[\sum_{i=1}^{\infty} f_1(p_1, t, i)\right]\right\rangle$$

$$f_5(p_5, t, i) = \langle 0, 0\rangle \quad \text{for} \quad i > 2.$$

Clearly $\Gamma_2$, $\Gamma_3$, $\Gamma_4$, and $\Gamma_5$ are all 2–dimensional systems of particle mechanics. By Theorem 7, the system $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4 + \Gamma_5$ is also a system of particle mechanics, and $\Gamma_1$ is a subsystem of $\Gamma$. Moreover, it is easily seen from the construction that $\Gamma$ is Newtonian.

From the proof of the above theorem we conclude that an $n$–dimensional system of particle mechanics containing $k$ particles can be embedded in a Newtonian system with $k(1 + 2^n)$ particles. This bound can in some cases be improved; thus it is easy to show that a 1–dimensional system containing $k$ particles can be embedded in a Newtonian system with $k + 1$ particles.

It is interesting to notice that Theorem 8 could not be proved if we adopted the Axiom of Impenetrability stated in Section 3. This is seen by considering any system containing just one particle, which is subjected to an infinity of non-vanishing forces, no two of which are collinear. In order to embed such a system in a Newtonian system satisfying the Axiom of Impenetrability, we should have to add, for each force, a particle lying along its line of action, which would violate Axiom P1.

Even assuming the Axiom of Impenetrability, however, it is possible to prove an embedding theorem if we generalize the notion of a Newtonian system by dropping the requirement that a pair of balanced forces lie along the line connecting the two particles;[12] if $\Gamma = \langle P, T, m, s, f\rangle$ is a system of particle mechanics, where $T$ is a finite closed interval, then $\Gamma$ can be embedded in a generalized Newtonian system $\Gamma'$; moreover, if $\Gamma$ satisfies the Axiom of Impenetrability, then $\Gamma'$ too can be taken to satisfy this axiom.

**5. Independence of Primitive Notions.** We now turn to the question of the independence of our primitive notions. As was mentioned in Section 1, the sets $P$ and $T$ are definable in terms of the functions $m$, $s$, and $f$; we shall show, on the other hand, that $m$, $s$, and $f$ are not definable in terms of the other primitives. This is done by the method of Padoa.[13] All the systems[14] employed are ultra-classical, and hence, *a fortiori*, Newtonian systems of particle mechanics—which shows that $m$, $f$, and $s$ would remain mutually independent even if we added Newton's Third Law to our set of axioms.

*Independence of m.* Let $P$ be the set whose only member is 1, and let $T$ be the set of all real numbers. Let

$$m_1(1) = 1$$

and

$$m_2(1) = 2.$$

For any $t$ in $T$, let

$$s(1, t) = \langle 1, 1, 1\rangle.$$

For any $t$ in $T$, and for any positive integer $i$, let

$$f(1, t, i) = \langle 0, 0, 0\rangle.$$

Then it is easily verified that $\langle P, T, m_1, s, f\rangle$ and $\langle P, T, m_2, s, f\rangle$ are both systems of particle mechanics (and indeed, ultra-classical systems of particle mechanics). This shows, however, that $m$ is not definable in terms of $P$, $T$, $s$, and $f$; for if $m$ were definable, then there could be only one $m$, for given $P$, $T$, $s$, and $f$, which would satisfy our axioms.

[12] Compare the discussion in Footnote 8.

[13] For an explanation of this method, see McKINSEY [1], PADOA [1], or TARSKI [1].

[14] It should be noticed that each of the six systems to follow gives a proof of the consistency of particle mechanics—or, properly speaking, a proof that particle mechanics is consistent if analysis is consistent. Here we use the notion of consistency in its usual logical sense (that the axioms do not imply a contradiction), not in the sense of HAMEL (see HAMEL [2], pp. 40–42), who thinks of a consistency problem as the problem of showing, for a given set of equations involving forces, that there exists a system of mechanics satisfying the equations (thus he uses the notion as it is used, for example, in algebra, when one calls a set of simultaneous equations "consistent" if they possess a solution).

*Independence of s.* Let $P = \{1, 2\}$, and let $T = [2, 3]$. Let

$$m(1) = m(2) = 1.$$

For every $t$ in $T$, let

$$s_1(1, t) = \langle t^2 - t, t, 4\rangle$$

$$s_1(2, t) = \langle -t^2 + t, t, 4\rangle,$$

and let

$$s_2(1, t) = \langle t^2, 0, 0\rangle$$

$$s_2(2, t) = \langle -t^2, 0, 0\rangle.$$

For every $t$ in $T$, let

$$f(1, t, 2) = \langle 2, 0, 0\rangle$$

$$f(2, t, 1) = \langle -2, 0, 0\rangle,$$

and for $i \neq 2$ and $j \neq 1$, let

$$f(1, t, i) = f(2, t, j) = \langle 0, 0, 0\rangle.$$

Then $\langle P, T, m, s_1, f\rangle$ and $\langle P, T, m, s_2, f\rangle$ are ultra-classical systems of particle mechanics.

*Independence of f.* Let $P = \{1, 2, 3\}$, and let $T = [1, 2]$. Let

$$m(1) = m(2) = m(3) = 1.$$

For all $t$ in $T$, let

$$s(1, t) = \langle t^2, 0, 0\rangle$$

$$s(2, t) = \langle -t^2, 0, 0\rangle$$

$$s(3, t) = \langle 0, 0, 0\rangle.$$

For all $t$ in $T$, let

$$f_1(1, t, 2) = \langle 2, 0, 0\rangle$$

$$f_1(1, t, i) = \langle 0, 0, 0\rangle \qquad \text{for } i \neq 2$$

$$f_1(2, t, 1) = \langle -2, 0, 0\rangle$$

$$f_1(2, t, i) = \langle 0, 0, 0\rangle \qquad \text{for } i \neq 1$$

$$f_1(3, t, i) = \langle 0, 0, 0\rangle \qquad \text{for all } i.$$

For all $t$ in $T$, let

$$\begin{aligned}
f_2(1, t, 2) &= \langle 1, 0, 0\rangle \\
f_2(1, t, 3) &= \langle 1, 0, 0\rangle \\
f_2(1, t, i) &= \langle 0, 0, 0\rangle && \text{for } i \notin \{2, 3\} \\
f_2(2, t, 1) &= \langle -1, 0, 0\rangle \\
f_2(2, t, 3) &= \langle -1, 0, 0\rangle \\
f_2(2, t, i) &= \langle 0, 0, 0\rangle && \text{for } i \notin \{1, 3\} \\
f_2(3, t, 1) &= \langle -1, 0, 0\rangle \\
f_2(3, t, 2) &= \langle 1, 0, 0\rangle \\
f_2(3, t, i) &= \langle 0, 0, 0\rangle && \text{for } i \notin \{1, 2\}.
\end{aligned}$$

Then $\langle P, T, m, f_1, s\rangle$ and $\langle P, T, m, f_2, s\rangle$ are (ultra-classical) systems of particle mechanics.

*Remarks.* All the systems described above in order to prove independence have a highly trivial character. Nevertheless, some of the results established (namely, that $m$ and $f$ are not definable) may seem a little surprising, in view of the fact that it has been rather generally held that mass and force are definable. We shall try to dispel misunderstanding by considering some of the definitions that have been proposed.

Kirchhoff (in Kirchhoff **[1]**) proposes that we define the force acting on a body simply as the product of the mass of the body by its acceleration;[15] this would not merely eliminate the primitive notion $f$, but would make it unnecessary to postulate Newton's Second Law. This procedure, however, would enable one to define only the resultant force acting on a particle—not the originally given forces. Thus, using the notation of this paper, if we set (for all $p$ in $P$ and $t$ in $T$) $R(p, t) = \sum_{i=1}^{\infty} f(p, t, i)$, then Kirchhoff's suggestion would provide a definition for the function $R$, but not for the function $f$. There is thus no contradiction between Kirchhoff's suggestion and our proof of the independence of $f$.

Mach[16] proposes to define the relative masses of two particles as the inverse ratio of their "mutually induced" accelerations when they are isolated from other particles. In our terms this means the following: suppose that $\langle P, T, m, s, f\rangle$

[15] HAMEL (in HAMEL **[3]**, p. 7) objects to this proposed definition on the ground: "Wäre sie richtig, dann wäre die Mechanik keine Naturwissenschaft mehr, sondern eine Tautologie!" It is clear, however, that even if KIRCHHOFF's definition were adopted, other axioms would be needed.

[16] See MACH **[1]**, pp. 264–277. For a more recent formulation of MACH's views on this subject, see LINDSAY & MARGENAU **[1]**, pp. 92–93.

is a Newtonian system of particle mechanics, where $P$ contains just two elements, say $p_1$ and $p_2$; then

$$m(p_1)\frac{d^2}{dt^2}s(p_1, t) = \sum_{i=1}^{\infty} f(p_1, t, i)$$

and

$$m(p_2)\frac{d^2}{dt^2}s(p_2, t) = \sum_{i=1}^{\infty} f(p_2, t, i);$$

since the system is Newtonian, however,

$$\sum_{i=1}^{\infty} f(p_1, t, i) = -\sum_{i=1}^{\infty} f(p_2, t, i),$$

and hence

$$m(p_1)\frac{d^2}{dt^2}s(p_1, t) = -m(p_2)\frac{d^2}{dt^2}s(p_2, t);$$

if $d^2/dt^2\, s(p_1, t)$ and $d^2/dt^2\, s(p_2, t)$ are not zero, therefore, the quantity $(m(p_1)/m(p_2))$ is equal to the ratio of the absolute values of $d^2/dt^2\, s(p_2, t)$ and $d^2/dt^2\, s(p_1, t)$.

With regard to this proposed definition, we notice at once that, in its present form, it applies only to Newtonian systems containing just two particles—and, indeed, even to such systems only under the hypothesis that the resultant forces are nor identically zero. Moreover, Pendse ([**1**], [**2**], [**3**])has shown that it is not possible to extend Mach's "definition" to Newtonian systems containing an arbitrary number of particles; in particular, Pendse shows that for more than seven particles, a knowledge of the "mutually induced" accelerations of the particles is not in general sufficient for a unique determination of the ratios of the masses of the particles.[17]

Mach's idea was probably that, when dealing with a given particle in a given system $\Gamma$, an experimenter could determine its mass by putting it into a Newtonian system of the kind described, and then put it back into $\Gamma$, with the assumption that the mass would remain invariant. In our terms, however, there is no way of carrying out these shiftings of a particle from one system to another; Mach's suggestion would allow a formal definition of mass only in case one were dealing with a greatly extended system, which would contain, among other things, the notion of an experimenter; it is not at all clear, however, that it would be possible to axiomatize such an extended system in a satisfactory way.

We should like to point out, in closing this discussion, that in order to be sure that one notion is independent of others it is not always necessary to know precisely what axioms are to be imposed on the notions: it is sufficient if one can feel

[17] In this connection, see SIMON [**1**].

certain that each of two models is a realization of whatever axioms one would want to impose. Thus if anyone were to hold, for example, that in particle mechanics mass is definable in terms of position and force, then he should maintain that not both the systems $\langle P, T, m_1, s, f\rangle$ and $\langle P, T, m_2, s, f\rangle$ of our independence proof are actually systems of particle mechanics; he would have to assume an axiom, in a word, which would rule out the possibility of a system consisting of a single particle, lying quietly at rest all by itself, acted on by no forces.

## References


[1] APPELL, P. *Traité de Mécanique Rationnelle*, Tome I, 6th ed., Paris: 1941.

[1] BANACH, S. *Mechanics*, translated by E. J. Scott. Warsaw, 1951.

[1] GOODMAN, NELSON. "The Problem of Counterfactual Conditionals", *Journal of Philosophy*, **44** (1947), pp. 113–128.

[1] GRAMMEL, R. "Kinetik der Massenpunkte", *Handbuch der Physik*, **5**, pp. 305–372.

[1] HERMES, HANS. Eine Axiomatisierung der allgemeinen Mechanik, Forschungen zur Logik und zur Grundlegung der exakten Wissenschaften, new series, **3**, S. Hirzel, Leipzig 1938, 48 pp.

[1] HAMEL, G. "Über die Grundlagen der Mechanik", *Mathematische Annalen*, **66** (1908), pp. 350–397.

[2] HAMEL, G. "Die Axiome der Mechanik", *Handbuch der Physik*, **5**, pp. 1–42.

[3] HAMEL, G. *Theoretische Mechanik*, Berlin 1949.

[1] HERTZ, H. *The Principles of Mechanics*, translated by D. E. JONES & J. T. WALLEY, London, 1899.

[1] JOOS, G. *Theoretical Physics*, translated by I. M. FREEMAN. New York, 1934.

[1] KIRCHHOFF, G. *Vorlesungen über mathematische Physik*, **1** Mechanik, 1878.

[1] LINDSAY, R. B., & MARGENAU, H. *Foundations of Physics*, New York, 1936.

[1] MACH, E. *Science of Mechanics*, translated from the German by T. J. MCCORMACK, 5th American edition, LaSalle, Ill. 1942.

[1] MARCOLONGO, R. *Theoretische Mechanik*, 2 vol., Deutsch von H. E. TIMERDING, Leipzig 1911 and 1912.

[1] MCKINSEY, J. C. C. "On the independence of undefined ideas", Bulletin of the *American Mathematical Society*, 1935, pp. 291–297.

[1] OSGOOD, W. F. *Mechanics*, New York, 1937.

[1] PADOA, ALESSANDRO. "Essai d'une théorie algébrique des nombres entiers, précédé d'une introduction logique à une théorie déductive quelconque", *Bibliothèque du Congrès International de Philosophie*, **3** (1900).

[1] PENDSE, C. G. "A note on the definition and determination of mass in Newtonian mechanics", *Philosophical Magazine* (7), **xxiv** (1937), pp. 1012–22.

[2] PENDSE, C. G. "A further note on the definition and determination of mass in Newtonian mechanics", *Ibid.*, **xxvii** (1939), pp. 51–61.

[3] PENDSE, C. G. "On mass and force in Newtonian mechanics", *Ibid.*, **xxix** (1940), pp. 477–484.

[1] QUINE, W. V. *Methods of Logic*. Henry Holt and Company, New York, 1950.

[1] ROSSER, BARKLEY. Review of HERMES [1], *Journal of Symbolic Logic*, **3** (1938), pp. 119–120.

[1] SIMON, H. A. "Axioms of Newtonian mechanics", *Philosophical Magazine* (7), **xxxvi** (1947), pp. 888–905.

[1] TARSKI, A. "Einige methodologische Untersuchungen über die Definierbarkeit der Begriffe," *Erkenntnis*, **5** (1935–36), pp. 80–100.