Basic Fluid Dynamics

Yue-Kin Tsang

February 9, 2011

1 Continuum hypothesis

In the continuum model of fluids, physical quantities are considered to be varying continuously in space, for example, we may speak of a velocity field $\vec{u}(\vec{x},t)$ or a temperature field $T(\vec{x},t)$. The "local" values of such quantities at a single point P in space should be understood as average values over a small region of size L_p about P. This averaging procedure is only meaningful if the region is large enough to contain many molecules and yet small when compared to the length scale of the macroscopic phenomena under consideration. Specifically, if L_m represents the molecular length scale ($\sim 10^{-9}$m) and L_f is the length scale of the fluid motion being studied, we assume that there exists a scale separation such that

$$L_m \ll L_p \ll L_f \,.$$

Figure 1 illustrates such scale separation and show how a "local" density at a point can be defined.

Within the continuum model, we define a fluid particle as a point which moves with the velocity of the fluid at that point. Hence the trajectory of a fluid particle $\vec{x}(t)$ is given by

$$\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = \vec{u}(\vec{x},t)\,. \tag{1}$$

2 Example: channel flow

We shall introduce several basic concepts using a simple example. We consider a fluid of constant density ρ_0 flowing between two large parallel planes separated by a distance of $2a$.

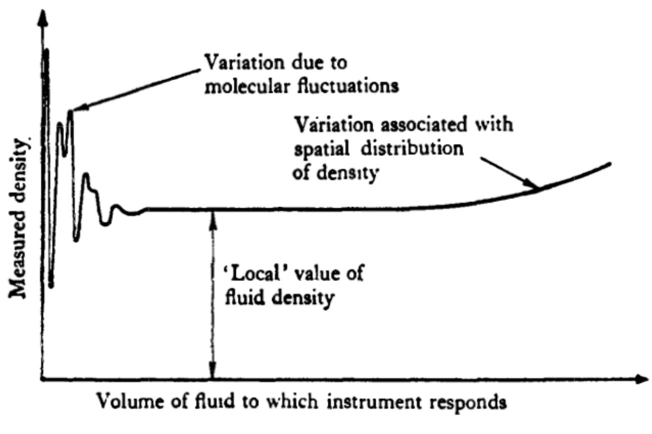

Figure 1: The continuum hypothesis (Batchelor, Introduction to Fluid Dynamics).

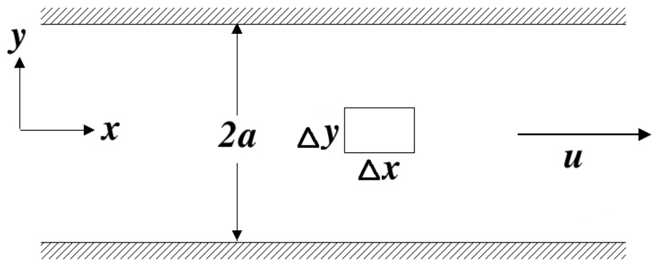

Figure 2: Channel flow. The coordinate system is setup such that $y = 0$ is at the middle of the channel and the z-direction is out of the page.

This configuration is called the channel flow. Pressures p_1 and p_2 are maintained at the ends of the channel. Figure 2 summarizes our notations.

Since the planes are "large", boundary effects in the x- and z-directions are ignored. From experimental observations, we know that when the velocity $\vec{u} = (u, v, w)$ is not too large, the flow is steady (i.e. $\partial/\partial t = 0$) and unidirectional, hence we take $v = w = 0$. It can be shown that $\vec{u}$ is independent of the streamwise coordinate for unidirectional flow (see Problem Set 1), hence, together with the symmetry in the z-direction, our goal is to determine the velocity profile $u(y)$.

Boundary conditions. We shall employ the *no-slip condition* at the walls, i.e. there is no relative tangential velocity between the walls and the fluid. For our present problem, it means

$$u(a) = u(-a) = 0\,. \tag{2}$$

The justification for the no-slip condition lies in experimental observations.

Forces acting on a fluid. We consider the following types of force:

1. *Body forces* (volume forces) are forces externally applied to a fluid and is part of the problem specification, they act on the whole volume of a fluid element.

2. *Surface forces* act across neighboring surfaces within a fluid and are of molecular origin. The surface force per unit area is called the local *stress.*

    (a) Pressure: stress acting normal to a surface.

    (b) Viscous stress: opposes relative movements between neighboring fluid particles.

We now consider the various forces acting on a fluid element of sides Δx, Δy and Δz in our channel flow problem, see Fig. 2. There is no body force acting on the fluid element, together with the assumption of steady unidirectional flow $v = w = 0$, we deduce that there is no pressure gradient in the y- and z-direction, hence $p = p(x)$ (see also Problem Set 1). The net pressure force on the fluid element is therefore

$$\begin{aligned}
&p(x)\Delta y\Delta z - p(x+dx)\Delta y\Delta z \\
=\,&p(x)\Delta y\Delta z - \left[p(x) + \Delta x\frac{\partial p}{\partial x}\right]\Delta y\Delta z \\
=\,&-\frac{\partial p}{\partial x}\,\Delta x\Delta y\Delta z\,.
\end{aligned} \tag{3}$$

Recall that viscous actions oppose relative movements, hence we expect viscous forces to act on the upper and lower surfaces of the fluid element shown in Fig. 2. To proceed, we need a relation between the viscous stress and the velocity.

Newtonian fluids. Throughout this course, we shall consider Newtonian fluids only, meaning that the viscous stress τ is proportional to the *velocity gradient*. For our current setup, this implies

$$\tau = \mu \frac{\partial u}{\partial y}\,, \tag{4}$$

where the proportionality constant μ is called the *viscosity*. Air and water are examples of Newtonian fluid while corn syrup and other polymer solutions are non-Newtonian. Using the linear relation Eq. (4), the net viscous force acting on the fluid element is

$$\begin{aligned} &-\mu \left.\frac{\partial u}{\partial y}\right|_{y} \Delta x \Delta z + \mu \left.\frac{\partial u}{\partial y}\right|_{y+dy} \Delta x \Delta z \\ &= \mu \frac{\partial^2 u}{\partial y^2} \Delta x \Delta y \Delta z\,. \end{aligned} \tag{5}$$

As the flow is assumed to be steady, the fluid element is not accelerating and there is no net force acting on it. Hence, we conclude there is a balance between the pressure force and the viscous force. From Eq. (3) and Eq. (5), we have

$$\frac{\partial p}{\partial x} = \mu \frac{\partial^2 u}{\partial y^2}\,. \tag{6}$$

There is one more simplification we can make for a unidirectional flow. Since the pressure gradient $\partial p/\partial x$ is responsible for the flow in the x-direction and we already know that $u = u(y)$ is independent of x, we expect $\partial p/\partial x = -G$ where G is a constant (see Problem Set 1 for a rigorous derivation). Finally, using the boundary conditions Eq. (2), we get

$$\frac{\mathrm{d}^2 u}{\mathrm{d}y^2} = -\frac{G}{\mu}\,, \tag{7}$$

$$u(y) = \frac{G}{2\mu}(a^2 - y^2)\,. \tag{8}$$

Flux. The amount of a quantity with density q transported through an infinitesimal area $\Delta\vec{S}$ in time Δt by a velocity field $\vec{u}$ is equal to the product of q and the infinitesimal volume $\Delta\vec{S} \cdot \vec{u}\,\Delta t$. Hence the net rate of transfer across a finite surface S is

$$\Phi = \int_S q\vec{u} \cdot \mathrm{d}\vec{S}\,. \tag{9}$$

Φ is sometimes called flux, it is a scalar and is the amount flowing through per unit time. This terminology is somewhat ambiguous, in some contexts, flux refers to $q\vec{u}$ which is a vector and is the amount flowing through per unit time per unit area. For our channel flow, we can compute the *mass flux* Φ_m through a square of sides $2a$ in the yz-plane as follow,

$$\Phi_m = \int_{-a}^{a} \int_{-a}^{a} \rho_0 u(y)\,\mathrm{d}y\,\mathrm{d}z = \frac{4G\rho_0 a^4}{3\mu}\,. \tag{10}$$

3 Continuity equation

We now derive the equation that expresses the *conservation of mass* for a fluid. Consider an arbitrary volume V bounded by the surface S in a fluid, the rate of mass flowing *out* of V through S is

$$\oint_S \rho\vec{u}\cdot \mathrm{d}\vec{S} = \int_V \nabla\cdot(\rho\vec{u})\,\mathrm{d}V\,, \tag{11}$$

where $\mathrm{d}\vec{S}$ is in the direction of the outward normal of S. On the other hand, the rate of change of mass inside V is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\,\mathrm{d}V\,. \tag{12}$$

The conservation of mass then implies

$$\int_V \left[\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{u})\right]\mathrm{d}V = 0\,. \tag{13}$$

Since Eq. (13) is true for any arbitrary volume, we have

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{u}) = 0\,. \tag{14}$$

4 The Navier-Stokes equation

To derive a form of Newton's second law, or the momentum equation, suitable for a fluid, our strategy is to follow a fluid particle along its trajectory and compute its rate of change of momentum. This leads to the concept of substantive derivative.

Substantive derivative. Generally, for a scalar quantity $T(x,y,z,t)$ varying with space and time, the change in T corresponds to small changes in the space and time coordinates is

$$\Delta T = \frac{\partial T}{\partial t}\Delta t + \frac{\partial T}{\partial x}\Delta x + \frac{\partial T}{\partial y}\Delta y + \frac{\partial T}{\partial z}\Delta z\,. \tag{15}$$

If we now choose Δx, Δy and Δz to be the components of a small displacement of a fluid particle over time Δt, then by the definition Eq. (1), the rate of change of T following the fluid particle, denoted by D/Dt, is given by

$$\frac{\mathrm{D}T}{\mathrm{D}t} = \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}\,. \tag{16}$$

The operator that gives the time rate of change along a fluid trajectory,

$$\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + \vec{u}\cdot\nabla\,, \tag{17}$$

is called the substantive derivative or the material derivative.

Applying Eq. (17) to the components of the velocity $\vec{u}(\vec{x},t)$ gives the rate of change of momentum *density* of a fluid particle as

$$\rho\frac{\partial\vec{u}}{\partial t} + \rho(\vec{u}\cdot\nabla)\vec{u}\,. \tag{18}$$

We now turn to the forces acting on a fluid particle. A rigorous derivation of the surface force term is out of the scope of this course. In particular, the full expression of the viscous stress is rather complicated. Instead, inspired by Eq. (5) from our channel flow problem, we quote that the viscous force per unit volume in the general case is

$$\mu\nabla^2\vec{u} \tag{19}$$

for a Newtonian fluid. Expression (19) is an approximation, a very good one in most cases, and becomes exact when μ is constant and $\nabla\cdot\vec{u}=0$. Similarly, from Eq. (3), we have the pressure force per unit volume equals

$$-\nabla p\,. \tag{20}$$

Finally, denoting the body force density by $\vec{f}$ and collecting everything, we have the Navier-Stokes equation,

$$\frac{\partial\vec{u}}{\partial t}+(\vec{u}\cdot\nabla)\vec{u}=-\frac{1}{\rho}\nabla p+\nu\nabla^2\vec{u}+\frac{\vec{f}}{\rho}\,, \tag{21}$$

where we have introduced the kinematic viscosity defined as $\nu\equiv\mu/\rho$. The nonlinear term $(\vec{u}\cdot\nabla)\vec{u}$ is called the inertial term.

5 Incompressible fluids

A fluid is incompressible if the density of a fluid particle is constant. In other words, the rate of change of $\rho(\vec{x},t)$ following a particle path is zero,

$$\frac{\mathrm{D}\rho}{\mathrm{D}t}=0\,. \tag{22}$$

The continuity equation Eq. (14) then reduces to

$$\nabla\cdot\vec{u}=0\,. \tag{23}$$

Condition for incompressibility. A special case for which Eq. (22) holds is when ρ is a constant. We now derive a condition under which ρ can be considered as approximately constant, in other words, the change in density $\Delta\rho$ is negligible, that is

$$\frac{\Delta\rho}{\rho}\ll 1\,. \tag{24}$$

For simplicity, we shall restrict ourselves to isothermal flows so that any change in density $\Delta\rho$ is due to pressure variation only.

Let us do some *order of magnitude estimations*. From the compressibility β of a fluid,

$$\beta\sim\frac{1}{V}\frac{\Delta V}{\Delta p}\sim\frac{1}{\rho}\frac{\Delta\rho}{\Delta p} \tag{25}$$

$$\Rightarrow\quad\frac{\Delta\rho}{\rho}\sim\beta\Delta p \tag{26}$$

Here, the symbol "$\sim$" means "of the same order". To estimate Δp, we refer to the Navier-Stokes equation Eq. (21). We assume the flow is steady or one in which temporal rate of change is small $\partial/\partial t=0$ and there is no body force $\vec{f}=0$. We choose to consider the case where the viscous term is small, hence there is a *dominant balance* between the pressure term and the inertial term,

$$(\vec{u}\cdot\nabla)\vec{u} \sim \frac{1}{\rho}\nabla p \tag{27}$$

$$\Rightarrow \quad \frac{U\,\Delta u}{L} \sim \frac{\Delta p}{\rho L}\,, \tag{28}$$

where U is a *typical* value of the velocity and L is a typical length scale in the problem. For velocity difference, we can make the estimate $\Delta u \sim U$. On the other hand, since only pressure difference is dynamically relevant, we have to keep Δp as it is. Hence, by Eq. (28), we get $\Delta p \sim \rho U^2$ and

$$\frac{\Delta\rho}{\rho} \sim \beta\rho U^2 \sim \frac{U^2}{c^2} = \mathrm{Ma}^2\,, \tag{29}$$

where $c^2 \sim 1/\beta\rho$ is the sound speed in the fluid and $\mathrm{Ma} \equiv U/c$ is called the Mach number. Therefore, we arrive at the conclusion that the flow can be considered as incompressible if $\mathrm{Ma} \ll 1$, that is the flow speed is much less than the sound speed in the fluid.

6 Energy budget

Consider a volume V in a fluid of constant density ρ_0 bounded by a closed surface S. The total kinetic energy in V is

$$K = \frac{\rho_0}{2}\int_V |\vec{u}|^2\,\mathrm{d}V\,. \tag{30}$$

An energy balance equation for the volume V can be obtained by taking the dot product of $\vec{u}$ and the Navier-Stokes equation and then integrating over V. Let us consider each term separately as follow.

(i)

$$\int_V \vec{u}\cdot\frac{\partial\vec{u}}{\partial t}\,\mathrm{d}V = \int_V \frac{1}{2}\frac{\partial|\vec{u}|^2}{\partial t}\,\mathrm{d}V = \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_V |\vec{u}|^2\,\mathrm{d}V\,. \tag{31}$$

(ii) Using the vector identity,

$$(\vec{u}\cdot\nabla)\vec{u} = (\nabla\times\vec{u})\times\vec{u} + \frac{1}{2}\nabla u^2\,, \tag{32}$$

and notice that $(\nabla\times\vec{u})\times\vec{u}$ is perpendicular to $\vec{u}$, we have

$$\int_V \vec{u}\cdot[(\vec{u}\cdot\nabla)\vec{u}]\,\mathrm{d}V = \frac{1}{2}\int_V \vec{u}\cdot\nabla u^2\,\mathrm{d}V = \frac{1}{2}\oint_S (u^2\,\vec{u})\cdot\mathrm{d}\vec{S}\,, \tag{33}$$

where we have use $\nabla\cdot\vec{u}$ in the last step.

(*A mathematical digression.* Instead of using the identity Eq. (32), we can also evaluate $\vec{u}\cdot[(\vec{u}\cdot\nabla)\vec{u}]$ as follow. We shall use the summation convention and the definition $\vec{A}\cdot\vec{B} = A_iB_i$. The ith component of the vector $(\vec{u}\cdot\nabla)\vec{u}$ is $(u_j\partial_j)u_i$, so

$$\begin{aligned}
\vec{u}\cdot[(\vec{u}\cdot\nabla)\vec{u}] &= u_i(u_j\partial_j u_i) = \partial_j(u_iu_ju_i) - u_i\partial_j(u_iu_j) \\
&= \partial_j(u_iu_ju_i) - u_iu_j\partial_j u_i \qquad (\because \partial_j u_j = 0) \\
\therefore \qquad 2u_iu_j\partial_j u_i &= \partial_j(u_iu_iu_j)\,.
\end{aligned} \tag{34}$$

Hence, $2\vec{u}\cdot[(\vec{u}\cdot\nabla)\vec{u}] = \nabla\cdot(u^2\vec{u})$ and Eq. (33) follows.)

(iii)

$$\int_V \vec{u}\cdot\nabla p\,\mathrm{d}V = \int_V \nabla\cdot(p\vec{u})\,\mathrm{d}V = \oint_S (p\vec{u})\cdot\mathrm{d}\vec{S}\,. \tag{35}$$

(iv) For the x-component of the velocity u,

$$\int_V u\nabla^2 u\,\mathrm{d}V = \int_V \left[\nabla\cdot(u\nabla u) - |\nabla u|^2\right]\,\mathrm{d}V = \oint_S u\nabla u\cdot\mathrm{d}\vec{S} - \int_V |\nabla u|^2\,\mathrm{d}V\,. \tag{36}$$

We have similar expressions for v and w.

(v)

$$\int_V (\vec{u}\cdot\vec{f}\,)\,\mathrm{d}V = \text{total power input into the volume } V \text{ by the body force.} \tag{37}$$

Collecting all these results, we have the energy budget for the volume V,

$$\begin{aligned}\frac{\mathrm{d}K}{\mathrm{d}t} = &-\oint_S \left(\frac{\rho_0 u^2}{2}\right)\vec{u}\cdot\mathrm{d}\vec{S}\\ &+\oint_S \left[-p\vec{u} + \mu(u\nabla u + v\nabla v + w\nabla w)\right]\cdot\mathrm{d}\vec{S}\\ &-\mu\int_V \left(|\nabla u|^2 + |\nabla v|^2 + |\nabla w|^2\right)\,\mathrm{d}V\\ &+\int_V (\vec{u}\cdot\vec{f}\,)\,\mathrm{d}V\,.\end{aligned} \tag{38}$$

The change in kinetic energy inside V results from the action of the terms on the right-hand side of Eq. (38). Comparing with Eq. (9), we recognize that the first term on the right is the rate of transfer of kinetic energy out of V through S. The second and third terms are the work done by the surface forces. Notice that the third term is non-negative and it represents the *dissipative* action of the viscosity. The final term is the power input by the externally applied body force.

7 Non-dimensionalization: the Reynolds number

We want to rewrite the Navier-Stokes equation in dimensionless form. The first step is to identify typical values of the various variables involved. These typical values depend on the problem configuration. We once again use the channel flow in Fig. 2 as an example:

variables	$\vec{x}$	t	$\vec{u}$	p	
typical values	L		U		from the problem configuration, L=a and U=$u(0)$
		L/U			no time scale provided by the problem specification
				ρU^2	see notes below

Note: The pressure difference cannot be varied independently of the velocity and its typical value is not known *a priori*. There are two dimensionally correct combinations for pressure: ρU^2 and $\mu U/L$. We arbitrarily choose ρU^2 and discuss the consequences of this choice later.

Define the dimensionless variables:

$$\vec{x}' = \frac{\vec{x}}{L},\quad t' = \frac{U}{L}t,\quad \vec{u}' = \frac{\vec{u}}{U},\quad p' = \frac{p}{\rho U^2} \tag{39}$$

and by substitution into the Navier-Stokes equation Eq. (21), we get

$$\frac{\partial \vec{u}'}{\partial t'} + (\vec{u}' \cdot \nabla')\vec{u}' = -\nabla' p' + \frac{1}{\mathrm{Re}} \nabla'^2 \vec{u}' \,, \tag{40}$$

where the Reynolds number Re is defines as,

$$\mathrm{Re} \equiv \frac{UL}{\nu} \,. \tag{41}$$

The Reynolds number measures the relative size of the inertial term to the viscous term,

$$\frac{(\vec{u} \cdot \nabla)\vec{u}}{\nu \nabla^2 \vec{u}} \sim \frac{U^2/L}{\nu U/L^2} = \mathrm{Re} \,. \tag{42}$$

The following are two reasons to consider the dimensionless form of the Navier-Stokes equation:

1. Consider flows that satisfy the same non-dimensional boundary conditions and initial conditions. If these flows have the same Re, then they are described by the same non-dimensional solution. Hence, by solving Eq. (40) for one value of Re, we obtain a family of infinitely many solutions. This is called *dynamical similarity.*

2. The relative magnitudes of the different terms in the equation are made explicit. For example, let us consider the case of steady flow. Note that by the definitions Eq. (39), the variables $\vec{x}'$ and $\vec{u}'$ are of order one $\mathcal{O}(1)$. Hence,

$$\underbrace{(\vec{u}' \cdot \nabla')\vec{u}'}_{\mathcal{O}(1)} = -\underbrace{\nabla' p'}_{?} + \frac{1}{\mathrm{Re}} \underbrace{\nabla'^2 \vec{u}'}_{\mathcal{O}(1)} \,. \tag{43}$$

   (a) High Reynolds number flow, $\mathrm{Re} \gg 1$:

$$(\vec{u}' \cdot \nabla')\vec{u}' = -\nabla' p' \,, \tag{44a}$$

$$(\vec{u} \cdot \nabla)\vec{u} = -\frac{1}{\rho} \nabla p \,. \tag{44b}$$

      i. The viscosity μ becomes unimportant and does not appear in Eq. (44).
      ii. Eq. (44) implies $\Delta p \sim \rho U^2$ and $\nabla' p' \sim \mathcal{O}(1)$. Hence in this case, our initial choice ρU^2 indeed represents typical pressure difference.
      iii. If we had defined $p' = p(L/\mu U)$ instead, we would get

$$(\vec{u}' \cdot \nabla')\vec{u}' = -\frac{1}{\mathrm{Re}} \nabla' p' + \frac{1}{\mathrm{Re}} \nabla'^2 \vec{u}' \,. \tag{45}$$

      We cannot ignore the first term on the right even $\mathrm{Re} \gg 1$ since $\nabla' p'$ is not of $\mathcal{O}(1)$, in fact, $\nabla' p' \sim \mathrm{Re}$.

   (b) Low Reynolds number flow, $\mathrm{Re} \ll 1$:

$$-\nabla' p' + \frac{1}{\mathrm{Re}} \nabla'^2 \vec{u}' = 0 \,, \tag{46a}$$

$$\nabla p = \mu \nabla^2 \vec{u} \,. \tag{46b}$$

      i. The density ρ becomes irrelevant in this case.
      ii. Eq. (46) implies $\Delta p \sim \mu U/L$ and $\nabla' p' \sim 1/\mathrm{Re} \gg \mathcal{O}(1)$.

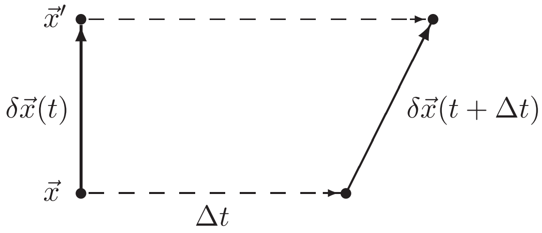

Figure 3: Evolution of a line element

8 Vorticity

The vorticity $\vec{\omega}$ is defined as the curl of the velocity field,

$$\vec{\omega} = \nabla \times \vec{u}\,. \tag{47}$$

Applying Stoke's theorem to a surface S bounded by a closed curve C, we have

$$\int_S \vec{\omega} \cdot \mathrm{d}\vec{S} = \oint_C \vec{u} \cdot \mathrm{d}\vec{\ell}\,, \tag{48}$$

indicating that the vorticity is related to rotational motions in a fluid. A flow with $\vec{\omega}$=0 everywhere is called an *irrotational* flow. We now show that the vorticity at a spatial point is two times the *local* angular velocity at that point.

We first derive an equation describing the time evolution of a line element $\delta\vec{x}$ in a velocity field $\vec{u}(\vec{x}, t)$. Referring to Fig. 3,

$$\vec{x}(t + \Delta t) = \vec{x}(t) + \vec{u}(\vec{x}, t)\Delta t\,, \tag{49}$$

$$\vec{x}'(t + \Delta t) = \vec{x}'(t) + \vec{u}(\vec{x}', t)\Delta t\,. \tag{50}$$

Taylor expanding $\vec{u}(\vec{x}', t)$ about $\vec{x}$,

$$\vec{u}(\vec{x}', t) = \vec{u}(\vec{x}, t) + [\delta\vec{x}(t) \cdot \nabla]\,\vec{u}(\vec{x}, t)\,. \tag{51}$$

Hence,

$$\delta\vec{x}(t + \Delta t) = \delta\vec{x}(t) + [\delta\vec{x}(t) \cdot \nabla]\,\vec{u}(\vec{x}, t)\Delta t\,, \tag{52}$$

$$\Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\delta\vec{x} = (\delta\vec{x} \cdot \nabla)\vec{u}\,. \tag{53}$$

To see how Eq. (53) gives us a physical interpretation of the vorticity, we decompose the velocity gradient tensor $\nabla\vec{u}$ into a symmetric part and an anti-symmetric part,

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta x_j = \delta x_i \partial_i u_j = \delta x_i \frac{\partial_i u_j + \partial_j u_i}{2} + \delta x_i \underbrace{\frac{\partial_i u_j - \partial_j u_i}{2}}_{r_{ij}}\,. \tag{54}$$

We identify that $r_{12} = \omega_3/2$, $r_{31} = \omega_2/2$ and $r_{23} = \omega_1/2$, i.e., $r_{ij} = \varepsilon_{ijk}\omega_k/2$ where ε_{ijk} is the Levi-Civita symbol. The last term in Eq. (54) becomes

$$\delta x_i r_{ij} = \frac{1}{2}\varepsilon_{jki}\omega_k \delta x_i = (\frac{\vec{\omega}}{2} \times \delta\vec{x})_j \tag{55}$$

and so we recognize $\mathrm{d}(\delta x_j)/\mathrm{d}t = \delta x_j r_{ij}$ describes a pure rotation of angular velocity $\vec{\omega}/2$.

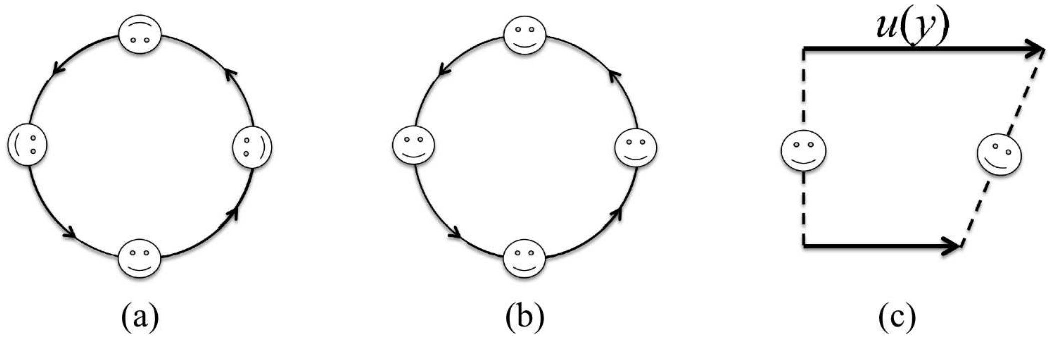

Figure 4: Fluid particles (a) moving in circles as well as spinning locally, $\vec{\omega} \neq 0$, (b) moving in circles with fixed orientations, $\vec{\omega} = 0$, (c) traveling in straight lines but spinning locally, $\vec{\omega} \neq 0$.

Example 1: *Rigid rotation* of angular velocity $\vec{\Omega} = \Omega \hat{k}$. In cylindrical coordinates (r, ϕ, z),

$$\vec{u} = \vec{\Omega} \times \vec{r} = r\Omega\,\hat{\phi}\,, \tag{56}$$

$$\vec{\omega} = 2\vec{\Omega}\,. \tag{57}$$

The vorticity equals twice the angular velocity everywhere as expected.

Example 2: *Line vortex* of strength $2\pi K$.

$$\vec{u} = \frac{K}{r}\hat{\phi}\,, \tag{58}$$

$$\omega_r = \omega_\phi = 0\,, \tag{59}$$

$$\omega_z = \frac{1}{r}\frac{\partial}{\partial r}\left(r \cdot \frac{K}{r}\right) = 0 \quad \text{if } r \neq 0 \tag{60}$$

$$\text{and} \quad \int_S \vec{\omega} \cdot \mathrm{d}\vec{S} = \oint_C \vec{u} \cdot \mathrm{d}\vec{\ell} = 2\pi K \tag{61}$$

for C being a circle centered at the origin. Hence, the vorticity is zero everywhere except on the axis.

Example 3: *Shearing* in the x-direction.

$$\vec{u} = u(y)\hat{i}\,, \tag{62}$$

$$\vec{\omega} = -\frac{\partial u}{\partial y}\,\hat{k} \neq 0\,. \tag{63}$$

The vorticity is non-zero even the trajectories of all fluid elements are straight lines.

These examples illustrate that vorticity represents local changes in the orientation of fluid particles, not their global motion in closed paths, see also Fig. 4. In fact, the definition of the vorticity makes no reference to an axis of rotation.

9 Vorticity equation

For a constant density fluid, hence $\nabla \cdot \vec{u} = 0$, taking the curl of the Navier-Stokes equation Eq. (21), we obtain the vorticity equation which describes the time evolution of the vorticity,

$$\frac{\mathrm{D}\vec{\omega}}{\mathrm{D}t} = (\vec{\omega} \cdot \nabla)\vec{u} + \nu\nabla^2\vec{\omega} + \vec{S}_\omega\,, \tag{64}$$

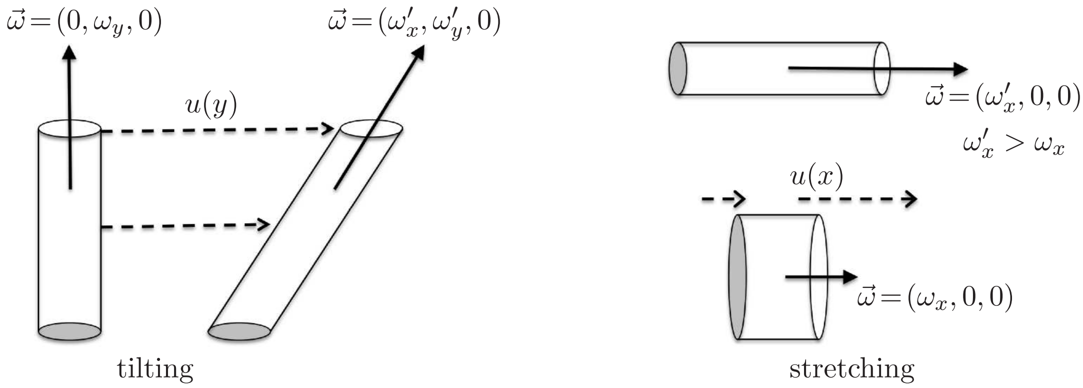

Figure 5: Vortex tilting (left) and vortex stretching (right).

where $\vec{S}_\omega \equiv \nabla \times (\vec{f}/\rho)$ is the vorticity source term. The term $\nu\nabla^2\vec{\omega}$ represents the diffusion of vorticity. To understand the physical meaning of the first term on the right of Eq. (64), let us consider the x-component vorticity ω_x and ignore diffusion and source,

$$\frac{\mathrm{D}\omega_x}{\mathrm{D}t} = \omega_x \frac{\partial u}{\partial x} + \omega_y \frac{\partial u}{\partial y} + \omega_z \frac{\partial u}{\partial z} \,. \tag{65}$$

Vortex tilting. The second term on the right of Eq. (65) has the following interpretation: as shown in Fig. 5, if $\vec{\omega}$ initially points in the y-direction, the velocity gradient $\partial u/\partial y$ "tilts" the vorticity vector in the xy-plane and thus generates an x-component ω_x in $\vec{\omega}$. Same goes for the last term in Eq. (65).

Vortex stretching. The first term on the right of Eq. (65) implies that as a fluid element is being stretched in the x-direction by the velocity gradient $\partial u/\partial x$, the x-component vorticity ω_x is amplified, see Fig. 5. This is because by the conservation of mass, the fluid element is contracting in the other two directions, hence its moment of inertia decreases while its angular velocity increases.

10 Circulation

The circulation Γ around a closed loop C is defined as,

$$\Gamma = \oint_C \vec{u} \cdot \mathrm{d}\vec{\ell}. \tag{66}$$

Kelvin's circulation theorem states that for inviscid flows with conservative body forces, i.e., ν=0 and $\vec{f} = -\nabla\phi$ in Eq. (21), circulation around a *material loop* is conserved:

$$\frac{\mathrm{D}}{\mathrm{D}t}\Gamma = \frac{\mathrm{D}}{\mathrm{D}t} \oint_S \vec{\omega} \cdot \mathrm{d}\vec{S} = 0 \,. \tag{67}$$

A consequence of Eq. (67) is that an inviscid irrotational flow remains irrotational. The proof of Kelvin's circulation theorem is as follow, for simplicity we assume constant density:

$$
\begin{aligned}
\frac{\mathrm{D}}{\mathrm{D}t}\oint_C \vec{u}\cdot \mathrm{d}\vec{\ell} &= \oint_C \frac{\mathrm{D}\vec{u}}{\mathrm{D}t}\cdot \mathrm{d}\vec{\ell} + \oint_C \vec{u}\cdot \frac{\mathrm{D}(\mathrm{d}\vec{\ell})}{\mathrm{D}t} \\
&= \frac{1}{\rho}\oint_C (-\nabla p + \vec{f})\cdot \mathrm{d}\vec{\ell} + \oint_C \vec{u}\cdot \mathrm{d}\vec{u} \\
&= -\frac{1}{\rho}\oint_C \nabla(p+\phi)\cdot \mathrm{d}\vec{\ell} + \frac{1}{2}\oint_C \mathrm{d}(u^2) \\
&= 0\,.
\end{aligned}
\tag{68}
$$

References

[1] D. J. Tritton, *Physical Fluid Dynamics* 2nd ed., Oxford University Press, 1988.

[2] G. K. Batchelor, *Introduction to Fluid Dynamics*, Cambridge University Press, 1967.