Glutty theories and the logic of antinomies

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1 Introduction

There are a variety of reasons why we would want a paraconsistent account of logic, that is, an account of logic where an inconsistent theory does not have every sentence as a consequence. One relatively standard motivation is epistemic in nature.[1] There is a high probability that we will come to hold inconsistent beliefs or inconsistent theories and we would like some account of how to reason from an inconsistent theory without everything crashing. Another motivation, rooted in the philosophy of logic or language, is that we want a proper account of entailment or relevant implication, where there is a natural sense in which inconsistent claims do not (relevantly) entail arbitrary propositions – where not every claim follows *from* arbitrary inconsistency.[2] A third motivation, the one which will occupy our attention here, is metaphysical or semantic. One might, for various reasons, endorse that there are 'true contradictions', or as they are sometimes called, truth-value *gluts* – true sentences of the form $\varphi \wedge \neg\varphi$, claims which are both true and false. We shall say that a *glut theorist* is one who endorses glutty theories – theories that are negation-inconsistent – with the full knowledge that they are glutty.

There are different kinds of metaphysical commitments that can lead one to be a glut theorist.

[1] For work in this tradition, see Rescher and Manor (1970); Schotch et al. (2009); Schotch and Jennings (1980).

[2] For work in this tradition, see Anderson and Belnap (1975); Anderson et al. (1992); Dunn and Restall (2002); Mares (2004); Slaney (2004).

One route towards glut theory arises from views about particular predicates of a language or the properties that those predicates express. Along these lines, a familiar route towards glut theory holds that certain predicates like 'is true', 'is a member of', or 'exemplifies' are *essentially* inconsistent: they cannot be (properly) interpreted in a way that avoids there being objects of which these predicates are both true and false. Such essentially glutty predicates – everywhere glutty with respect to something if glutty anywhere with respect to anything – are *antinomic*, as we shall say. Of course, one need not hold that a predicate is essentially inconsistent to think that it can give rise to gluts: there may be only contingently glutty predicates. For some predicates, whether they are properly interpreted consistently or inconsistently may depend on facts about the world. Priest, for example, has suggested that predicates like 'is legal' and 'has the right to vote' are of this sort Priest (2006). Acceptance of either sort of (essentially or contingently) inconsistent predicates is sufficient for being a glut theorist – though not necessary.

Another path one might take towards being a glut theorist is inevitable ignorance about the exact source of gluttiness. One might think that our best – and true – theory of the world will inevitably be inconsistent, even though we might, for all we know, remain ignorant of the source of the inevitable inconsistency. Indeed, one might have reason to be agnostic about the source of gluttiness: one is convinced that our best theory of the world (including truth, exemplification, sets, computability, modality, whatever) will be inconsistent, though also convinced that we will never be in good position to pinpoint the exact source of the inconsistency. Agnosticism about the particular predicates responsible for gluttiness remains an option for the glut theorist.

The question that arises is: how do our metaphysical commitments inform our choice of logic? We cannot ask this question without attending to the difference between formal and material consequence. Briefly, a logic takes a *material* approach to consequence when it builds in facts about the meaning of predicates, the properties they express, or the objects those predicates are about. A logic takes a *formal* approach to consequence when it abstracts away from all of these concerns. There are various ways a logic could be said to 'build in' such facts, and one of our aims below is to explore these in the context of metaphysical commitments to gluts. We carry out our discussion via a comparison of two paraconsistent logics, namely, the logic of paradox (LP) and the logic of antinomies (LA). The former is well-known in philosophy, discussed explicitly and widely by Priest (1979, 2006);[3] the latter is a closely related but far less familiar and equally less explored approach in philosophy, an approach advanced by Asenjo and Tamburino (1975).[4]

Below, we consider various philosophical motivations that could explain the logical differences between LA and LP. We shall argue that LA reflects a fairly distinctive set of metaphysical and philosophical commitments, whereas LP, like any formal logic, is compatible with a broad set of metaphysical and philosophical commitments. We illustrate these points below.

The discussion is structured as follows. §§2–3 present the target logics in terms of familiar model theory. §4 discusses the main logical differences in terms of differences in philosophical focus and metaphysical commitment. §5 closes by discussing the issue of detachment.

2 The Logic of Antinomies

The logic of antinomies (LA) begins with a standard first-order syntax. The logical vocabulary is $\vee, \neg, \forall$. Constants $c_0, c_1, \ldots$ and variables $x_0, x_1, \ldots$ are the only terms. The set $\mathbb{P}$ of predicate symbols is the union of two disjoint sets of standard predicate symbols: $\mathbb{A} = \{A_0, A_1, \ldots\}$ and $\mathbb{B} = \{B_0, B_1, \ldots\}$. (Intuitively, $\mathbb{A}$ contains the essentially classical predicates and $\mathbb{B}$ the essentially non-classical, essentially glutty predicates.) The standard recursive treatment defines the set of

[3]LP is the gap-free extension of FDE, the logic of tautological entailments; it is the dual of the familiar glut-free extension of FDE called ‘strong Kleene’ or ‘K3’. See Dunn (1966, 1976), Anderson and Belnap (1975), and Anderson et al. (1992).

[4]For purposes of accommodating glutty theories, the propositional logic LP was first advanced in Asenjo (1966) under the name *calculus of antinomies*; it was later advanced, for the same purpose, under the name ‘logic of paradox’ by Priest (1979), who also gave the first-order logic under the same name (viz., LP). What we are calling ‘LA’ is the first-order (conditional-free) logic advanced by Asenjo and Tamburino (1975), which was intended by them to be a first-order extension of Asenjo’s basic propositional logic. Due to what we call the LA predicate restriction (see page 4), LA isn’t a simple first-order extension of Asenjo’s propositional LP – as will be apparent below (see §4).

sentences.[5]

An LA interpretation I consists of a non-empty domain D, a denotation function d, and a variable assignment v, such that:

- for any constant c, $d(c) \in D$,
- for any variable x, $v(x) \in D$,
- for any predicate P, $d(P) = \langle P^+, P^- \rangle$, where $P^+ \cup P^- = D$.

The only difference from the standard LP treatment appears here, in the form of a restriction that captures the distinction between the antinomic (i.e., essentially glutty) and essentially classical predicates:

LA Predicate Restriction. *For any predicate P:*

- *if P is in $\mathbb{A}$, then the intersection $P^+ \cap P^-$ must be empty;*
- *if P is in $\mathbb{B}$, then the intersection $P^+ \cap P^-$ must be non-empty.*

As above, the A_is are the essentially classical predicates, while the B_is are those which are antinomic.[6]

$|\varphi|_v$ is the semantic value of a sentence φ with respect to a variable assignment v, which is defined in the standard recursive fashion. (We leave the relevant interpretation implicit, as it will always be obvious.) For atomics:

$$|Pt|_v = \begin{cases} 0 \text{ if } I(t) \notin P^+ \text{ and } I(t) \in P^- \\ 1 \text{ if } I(t) \in P^+ \text{ and } I(t) \notin P^- \\ \frac{1}{2} \text{ otherwise.} \end{cases}$$

[5]For simplicity, we focus entirely on unary predicates. Both LA and LP cover predicates of any arity, but focusing only on the unary case suffices for our purposes.

[6]The presentation in Asenjo and Tamburino (1975) is rather different; but we present their account in a way that affords clear comparison with LP.

The inductive clauses are as follows:

1. $|\varphi \vee \psi|_v = \max\{|\varphi|_v, |\psi|_v\}$.

2. $|\neg\varphi|_v = 1 - |\varphi|_v$.

3. $|\forall x\varphi|_v = \min\{|\varphi|_{v'} : v'$ is an x-variant of $v\}$.

Conjunction and existential quantification can be defined from these in the normal way.

LA consequence $\vdash_{LA}$ is defined as preservation of designated value, where the designated values are 1 and $\frac{1}{2}$. Thus, $\Gamma \vdash_{LA} \varphi$ holds (i.e., Γ implies/entails φ according to LA) if and only if no LA interpretation designates everything in Γ and fails to designate φ.

3 The Logic of Paradox

We obtain the logic LP simply by dropping the LA predicate restriction, but leaving all else the same. Thus, for purposes of 'semantics' or model theory of LP, there's no difference between $\mathbb{A}$-predicates and $\mathbb{B}$-predicates: they're all treated the same.

4 Contrast: LA and LP

We begin with formal contrast. While both logics are paraconsistent (just let $|\varphi|_v = \frac{1}{2}$ and $|\psi|_v = 0$, for at least some formalæ φ and ψ), there are some obvious but noteworthy formal differences between the logics LA and LP. LP permits the existence of a maximally paradoxical object – an object of which *every* predicate is both true and false – whereas LA does not. Indeed, LA – but not LP – validates 'explosion' for certain contradictions; for example, for any A_i in $\mathbb{A}$ and any φ,

$$A_i t \wedge \neg A_i t \vdash_{LA} \varphi.$$

Similarly, LA validates the parallel instances of detachment (modus ponens):

$$A_i t, A_i t \supset \varphi \vdash_{LA} \varphi$$

where $\varphi \supset \psi$ is defined as usual as $\neg\varphi \vee \psi$. But LP is different: not even a restricted version of detachment is available Beall et al. (2013).

As a final and nicely illustrative example, LA validates some existential claims that go beyond those involved in classical logic (e.g., $\exists x(\varphi \vee \neg\varphi)$, etc.), whereas LP does not. To see this, note that, for any B_i in $\mathbb{B}$, the following is a theorem of LA:

$$\exists x B_i x.$$

Since any predicate B_i in $\mathbb{B}$ must have at least some object in the intersection of its extension and antiextension, it follows that something is in its extension.

4.1 Metaphysics, Formal and Material Consequence

How are we to understand the logical differences between LA and LP? For a first pass, they might be naturally understood as arising from different notions of consequence: namely, material and formal consequence. The distinction may not be perfectly precise, but it is familiar enough.[7]

Material consequence relies on the 'matter' or 'content' of claims, while formal consequence abstracts away from such content. Example: there is no possibility in which 'Max is a cat' is true but 'Max is an animal' is not true; the former entails the latter if we hold the meaning – the matter, the content – of the actual claims fixed. But the given entailment fails if we abstract away from matter (content), and concentrate just on the standard first-order form: *Cm* does not entail *Am*.

The notion of formal consequence delivers conclusions based on logical form alone. Material

[7] See Read (1994, Ch. 2) wherein Read provides a defense of material consequence as logical consequence, and also for further references.

consequence essentially requires use of the content of the claims or the meaning of things like predicates that appear in them.

One way to understand Asenjo and Tamburino's proposal is that it gives a material consequence relation of a language arising from certain metaphysical commitments. It is clear that the logic reflects an assumption that certain predicates are essentially classical, and other predicates are essentially glutty – antinomic, as we have said. On this interpretation, the incorporation of essentially classical predicates reflects a metaphysical commitment that gluts cannot arise absolutely anywhere. Similarly, the semantic restriction on the B_i predicates reflects a metaphysical commitment that certain predicates, in virtue of their meaning, or the properties they express, must give rise to gluts: there is bound to be at least some object of which B_i is both true and false. Beall (2009) gives such a view: inconsistency unavoidably arises in the presence of semantic predicates like 'is true'. The typical semantic paradoxes like the liar require an inconsistent interpretation of the truth predicate, but this is compatible with the commitment to the essential classicality of all predicates in the truth-free fragment of the language.

But what if you wanted to give the *formal* consequence relation of a language that is motivated by Asenjo and Tamburino's metaphysical commitments? LP, we suggest, provides the formal consequence relation of such a language – abstracting from the matter or content to mere form. LA's predicate restriction is not a purely formal matter: that a predicate is either antinomic or essentially classical depends on its meaning. If we ignore content, and focus just on purely formal features of sentences, LA's predicate restriction falls away as unmotivated. And that's precisely what happens in LP: if we abstract away to 'pure form' then the content of predicates doesn't matter.

We observe that one might be concerned with material consequence, and yet still be motivated to adopt LP rather than LA. Suppose that all predicates are on par with respect to (in-) consistency: each might be glutty with respect to something – or not glutty at all. If one held this commitment, then LA's predicate restriction is inappropriate, or at least unmotivated. Indeed, even one predicate which is either contingently consistent or contingently inconsistent arrests the motivation for LA's predicate restriction. And one might think that ordinary cases of such predicates are not hard to find. Priest (2006, Ch. 13), for example, discusses such cases arising from considerations of the law. Suppose that we had laws that *all citizens have a right to vote* and *no felons have a right to vote*. It is then a contingent matter whether or not there are any gluts about rights to vote; it depends on whether anyone commits any felonies, and whether or not anything is classified as a felony. And you might hold a view wherein all predicates are like that: potentially glutty, one and all, but none antinomic – none essentially glutty.

There is another metaphysical route to LP. We might not start with any commitments about the nature of any predicates, their meaning, the properties they express, and whether or not they are essentially inconsistent. One might start with the commitment that one's theory is both true and inconsistent, while remaining agnostic about where to locate the origins of the inconsistency. There is no reason to think that this position excludes a material approach to consequence. It's just that such a view lacks any particular metaphysical commitments that would motivate a restriction on predicates like the LA predicate restriction.

Of course, from the material point of view, LA and LP far from exhaust the possibilities. So far we've mentioned fairly strong, all-or-nothing approaches. On a material approach to consequence, the proponent of LA is committed to all predicates being essentially classical or glutty, while the proponent of LP is committed to all predicates being potentially classical or glutty. Mixed approaches are available. These are achieved by adding obvious combinations to the LA predicate restriction – for example, some antinomic, some essentially classical, some neither, etc. We leave these to the reader for exploration.

We turn (briefly) to an issue peculiar to the logics under discussion: detachment or modus ponens.

5 Detachment

A salient problem for LP is that there is no detachable (no modus-ponens-satisfying) conditional definable in the logic Beall et al. (2013); and thus, historically, LP has been viewed as unacceptably weak for just that reason. A lesson one might try to draw from the above observations is that LP can be improved by shifting focus to the material notion of consequence. But this is not quite right. Though one fragment of LA differs from LP in that it satisfies detachment, LA is like LP in that detachment doesn't hold generally: arguments from φ and $\varphi \supset \psi$ to ψ have counterexamples.

On this score, Asenjo and Tamburino (1975), along with Priest (1979, 2006), have a solution in mind. The remedy is to add logical resources to the base framework to overcome such non-detachment.[8] But the remedy offered by Asenjo and Tamburino doesn't work, as we now briefly indicate.

Asenjo and Tamburino define a conditional $\rightarrow$ that detaches (i.e., φ and $\varphi \rightarrow \psi$ jointly imply ψ). The conditional is intended to serve the ultimate purpose of the logic, namely, to accommodate paradoxes in non-trivial theories (e.g., theories of naïve sets), and is defined thus:

$$|\varphi \rightarrow \psi|_v = \begin{cases} 0, & \text{if } |\psi|_v = 0 \text{ and } |\varphi|_v \in \left\{\frac{1}{2}, 1\right\} \\ \frac{1}{2}, & \text{if } |\psi|_v = \frac{1}{2} \text{ and } |\varphi|_v \in \left\{\frac{1}{2}, 1\right\} \\ 1, & \text{otherwise.} \end{cases}$$

The resulting logic, which we call $\text{LA}^{\rightarrow}$, enjoys a detachable conditional. In particular, defining $\vdash_{LA^{\rightarrow}}$ as above (no interpretation designates the premise set without designating the conclusion), we have:

$$\varphi, \varphi \rightarrow \psi \vdash_{LA^{\rightarrow}} \psi.$$

[8] Until very recently, Beall (2013), all LP-based glut theorists focused their efforts on the given task: adding logical resources to the base LP framework to overcome its non-detachment. Whether this is the appropriate response to the non-detachment of LP is something we leave open here.

The trouble, however, comes from Curry's paradox. Focusing on the set-theoretic version (though the truth-theoretic version is the same), Meyer et al. (1979) showed that, assuming standard structural rules (which are in place in LP and $\mathrm{LA}^{\rightarrow}$ and many other logics under discussion), if a conditional detaches and also satisfies 'absorption' in the form

$$\varphi \rightarrow (\varphi \rightarrow \psi) \vdash \varphi \rightarrow \psi$$

then the given conditional is not suitable for underwriting naïve foundational principles. In particular, in the set-theory case, consider the set

$$\mathsf{c} = \{x : x \in x \rightarrow \bot\}$$

which is supposed to be allowed in the Asenjo and Tamburino (and virtually all other) paraconsistent set theories.[9] By unrestricted comprehension (using the new conditional, which is brought in for just that job), where $\leftrightarrow$ is defined from $\rightarrow$ and $\wedge$ as per usual, we have

$$\mathsf{c} \in \mathsf{c} \leftrightarrow (\mathsf{c} \in \mathsf{c} \rightarrow \bot).$$

But, now, since the Asenjo–Tamburino arrow satisfies the given absorption rule, we quickly get

$$\mathsf{c} \in \mathsf{c} \rightarrow \bot$$

which, by unrestricted comprehension, is sufficient for c's being in c, and so

$$\mathsf{c} \in \mathsf{c}.$$

But the Asenjo–Tamburino arrow detaches: we get $\bot$, utter absurdity.

[9] Throughout, $\bot$ is 'explosive' (i.e., implies all sentences).

The upshot is that while LA may well be sufficient for standard first-order connectives, the 'remedy' for non-detachment (viz., moving to $\text{LA}^{\rightarrow}$) is not viable: it leads to absurdity.[10] Other LP-based theorists, notably Priest (1980) and subsequently Beall (2009), have responded to the non-detachability of LP by invoking 'intensional' or 'worlds' or otherwise 'non-value-functional' approaches to suitable (detachable) conditionals. We leave the fate of these approaches for future debate.[11]

6 Closing remarks

Philosophy, over the last decade, has seen increasing interest in paraconsistent approaches to familiar paradox. One of the most popular approaches is also one of the best known: namely, the LP-based approach championed by Priest. Our aim in this paper has been twofold: namely, to highlight an important predecessor of LP, namely, the LA-based approach championed first by Asenjo and Tamburino, and to highlight the salient differences in the logics. We've argued that the differences in logic reflect a difference in both background philosophy of logic and background metaphysics. LA is motivated by a material approach to logical consequence combined with a metaphysical position involving antinomic predicates, while LP is compatible with both a formal and material approach to consequence and can be combined with a large host of metaphysical commitments (including few such commitments at all).[12]

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[10] We note that Asenjo himself noticed this, though he left the above details implicit. We have not belabored the details here, but it is important to have the problem explicitly sketched.

[11] We note, however, that Beall has recently rejected the program of finding detachable conditionals for LP, and instead defends the viability of a fully non-detachable approach Beall (2013), but we leave this for other discussion.

[12] We note that Priest's ultimate rejection of LP in favor of his non-monotonic LPm (elsewhere called 'MiLP') reflects a move 'back' in the direction of the original Asenjo–Tamburino approach, where one has 'restricted detachment' and the like, though the latter logic (viz., LA) is monotonic. We leave further comparison for future debate. For some background discussion, see Priest (2006, Ch. 16) and Beall (2012) for discussion.

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